ON THE ROSENTHAL COMPACTA AND ANALYTIC SETS

ADAM KRAWCZYK

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ABSTRACT. We consider pointwise convergence in a separable Rosenthal compactum. The main result is that if $X \subset \mathbf{R}^{\omega^\omega}$ is a Rosenthal compactum, $Y \subset X$ is countable dense and $x \in X$, then the following are equivalent :

(i) Y has a countable base at x.

(ii) $\{(y_i) \in Y^\omega : \lim_{i \to \infty} y_i = x\}$ is analytic, when Y has the discrete topology.

1. INTRODUCTION

In the paper we will deal mainly with analytic spaces, i.e., continuous images of the Baire space. The class of analytic sets will be denoted by Σ_1^1. When X, Y are topological spaces then, throughout this paper, X^Y is the space of all functions from Y to X with pointwise convergence topology. We will follow the standard set theoretical notation, so an ordinal is equal to the set of its predecessors. In particular, the set of natural numbers will be denoted by ω, and if $n \in \omega$, then $n = \{0, \dots, n-1\}$. The only topology on ω which will be considered is the discrete one, so ω^ω is the Baire space.

Among compact subsets of $\mathbf{R}^{\omega^\omega}$ particularly nice are Rosenthal compacta, i.e., compact subsets $X \subset \mathbf{R}^{\omega^\omega}$ such that every element of X is a pointwise limit of continuous functions (see [R]). There are separable and not first countable Rosenthal compacta. Many examples of such compacta are constructed in [Ma]. However in the beautiful paper [BFT] Bourgain, Fremlin, and Talagrand proved that a Rosenthal compactum is a Frechet space.

The key to our paper is the notion of bisequentiality (see [P1, Definition 3.D.1]).

Definition 1. *The space X is bisequential at a point $x \in X$ if every ultrafilter p in X converging to x contains a sequence of sets converging to x.*

Pol in [P1] improved the result of Bourgain, Fremlin, Talagrand and obtained the following:

Theorem 1 (Pol). *A separable Rosenthal compactum is bisequential.*

The main theorem we are going to prove is the following:

Theorem 2. *Assume that X is a Rosenthal compactum, $Y \subset X$ is countable and dense, $x \in X$. Then the following are equivalent :*

   (i) *Y has a countable base at x.*
   (ii) *$\{(y_i) \in Y^\omega : \lim_{i \to \infty} y_i = x\}$ is $\mathbf{F}_{\sigma\delta}$.*
   (iii) *$\{(y_i) \in Y^\omega : \lim_{i \to \infty} y_i = x\}$ is analytic.*

In (ii) *and* (iii), *Y is considered with the discrete topology.*

It has to be pointed out that the set $\{(y_i) \in Y^\omega : \lim_{i \to \infty} y_i = x\}$ is always coanalytic. One can check it easily by direct computation, as in the proof of the theorem in §3.

The paper is organized as follows: In §2 we will prove three technical lemmas, which allow us derive in §3 the theorem and other corollaries.

2. LEMMAS

In this section we will work with families of subsets of ω or $\omega^{<\omega} = \bigcup_{n \in \omega} \omega^n$. We present notation and definitions for ω, but all can be applied to $\omega^{<\omega}$ as well.

In the theory of filters on ω there is a strong trend to denote everything by $*$. We will follow that nasty tendency: so $A \subset^* B$ means $A \setminus B$ is finite; $A =^* B$ stands for $A \subset^* B$ and $B \subset^* A$; $\omega^* = \beta\omega \setminus \omega$ is the set of non-principal ultrafilters on ω; but if $\mathscr{F} \subset \mathscr{P}(\omega)$ then $\mathscr{F}^* = \{\omega \setminus A : A \in \mathscr{F}\}$. Hence if $p \in \omega^*$ then p^* is the maximal ideal on ω dual to p, $p^{**} = p$ and $\forall \mathscr{F} \in \mathscr{P}(\omega) \exists p \in \omega^* \mathscr{F}^* \subset^* p^*$ is the following (false) statement: For every family $\mathscr{F}$ of subsets of ω there exists I—a maximal nonprincipal ideal on ω such that $\{B : \omega \setminus B \in \mathscr{F}\} \setminus I$ is finite.

Definition 2. *Let $\mathscr{F}, \mathscr{G} \subset \mathscr{P}(\omega)$. Then*

- $\widehat{\mathscr{F}} = \{B \subset \omega : \forall A \in \mathscr{F} \; B \cap A =^* \emptyset\}$.
- *$\mathscr{F}$ is countably $\mathscr{G}$-generated if there exists $\{A_n : n \in \omega\} \subset \mathscr{G}$ such that $B \subset A_0 \cup \cdots \cup A_k$ for all $B \in \mathscr{F}$, and some $k \in \omega$.*
- *$\mathscr{F}$ is bisequential if for every $p \in \beta\omega$ with $\mathscr{F} \subset p^*$, $\mathscr{F}$ is countably p^*-generated.*

Whenever $\mathscr{F}$ is an ideal then the definition of bisequentiality is closely connected with that from the introduction. Details become transparent in the next section.

The following lemma contains the key combinatorial part of the argument.

Lemma 1. *Assume $I \subset \mathscr{P}(\omega)$ is bisequential and $\mathscr{F} \subset I$ is not countably I-generated. Then there exists a pairwise disjoint family $\mathscr{A} = \{A_n : n \in \omega\}$ such that I is $\mathscr{A}$-generated and $\mathscr{F} \cap \mathscr{P}(A_n)$ is not countably I-generated for every $n \in \omega$.*

Proof. Consider $I' = \{C \subset \omega : \mathscr{F} \cap \mathscr{P}(C)$ is countably I-generated $\}$. By definition $I \subset I'$, and by assumption that $\mathscr{F}$ is not countably I-generated we get $\omega \notin I'$. It is easy to see that I' is an ideal. Let $p \in \beta\omega$ be such that $I' \subset p^*$.

Then, by bisequentiality of I we can find a family $\{D_n : n \in \omega\} \subset p^*$, such that I is $\{D_n : n \in \omega\}$-generated. Without loss of generality one can assume that D_n's are pairwise disjoint. Put $N = \{n \in \omega : D_n \in I'\}$.

Claim 1. $\bigcup_{n \in N} D_n \in I'$.

Proof of the Claim. By definition of I' for every $n \in N$ there exists $\{E_k^n : k \in \omega\} \subset I$, such that $\mathscr{F} \cap \mathscr{P}(D_n)$ is $\{E_k^n : k \in \omega\}$-generated. But then $\mathscr{F} \cap \mathscr{P}(\bigcup_{n \in N} D_n)$ is $(\bigcup_{n \in N} \{E_k^n : k \in \omega\})$-generated.

Indeed, let $C \in \mathscr{F} \cap \mathscr{P}(\bigcup_{n \in N} D_n)$. Then there exists $m \in \omega$ such that $C \subset D_0 \cup \cdots \cup D_{m-1}$. Since $C \subset \bigcup_{n \in N} D_n$ and D_n's are disjoint, $C \subset \bigcup_{n \in N \cap m} D_n$. Now, for every $n \in m \cap D$, we can find $k_n \in \omega$ such that $C \cap D_n \subset E_0^n \cup \cdots \cup E_{k_n}^n$; hence $C \subset \bigcup_{n \in N \cap m} E_0^n \cup \cdots \cup E_{k_n}^n$ and we are done. □

We will show now that $\omega \setminus N$ is infinite. Assume for a contradiction that it is not the case. Then $\bigcup_{n \notin N} D_n \in p^*$, $\bigcup_{n \in N} D_n \in I' \subset p^*$ and, by properties of D_n's and definition of I', we get $(\omega \setminus \bigcup_{n \in \omega} D_n) \subset (\omega \setminus \bigcup \mathscr{F}) \in I' \subset p^*$. This implies $\omega \in p^*$, a contradiction.

Now let $\{k_n : n \in \omega\}$ be a 1-1 enumeration of $\omega \setminus N$. Put $A_0 = \bigcup_{n \in N} D_n \cup D_{k_0}$, $A_n = D_{k_n}$ for $n > 0$. Obviously $\{A_n : n \in \omega\}$ satisfies the conclusion of the lemma. □

For $s \in \omega^{<\omega}$ let $\hat{s}$ denote the set $\{t \in \omega^{<\omega} : s \subset t\}$. Lemma 2 contains the topological part of the argument.

Lemma 2. *Assume* $I \in \Sigma_1^1$ *is a bisequential and not countably I-generated ideal on ω. Then there exists* $T : \omega^{<\omega} \xrightarrow{1-1} \omega$ *such that for* $J = T^{-1}(I)$ *the following holds*:

(i) $\forall f \in \omega \{f|n : n \in \omega\} \in J$.
(ii) $\forall B \in J \, \forall n \in \omega \, \exists s_0, \ldots, s_k \in \omega^n \, B \subset^* \hat{s}_0 \cup \cdots \cup \hat{s}_k$.

Proof. Let $\varphi : \omega^\omega \xrightarrow{\text{onto}} I$ be continuous. By induction on length of s (lh$s$) we construct F_s and A_s such that when $T(s) = \min A_s$, the following conditions will hold:

(a) $s \supset t \Rightarrow A_s \subset A_t \setminus \{T(t)\} \subset \omega$.
(b) $s, t \in \omega^n$, $s \neq t \Rightarrow A_s \cap A_t = \emptyset$.
(c) F_s is a closed subset of ω^ω.
(d) The diameter of F_s is less than $2^{-\text{lh} s}$.
(e) $s \supset t \Rightarrow F_s \subset F_t$.
(f) $\varphi(F_s) \subset \mathscr{F}_{\hat{s}}$, where $B \in \mathscr{F}_s$ if and only if $\{T(s|0), \ldots, T(s|(\text{lh} s - 1))\} \subset B \subset A_s \cup \{T(s|0), \ldots, T(s|(\text{lh} s - 1))\}$.
(g) $\varphi(F_s)$ is not countably I-generated.
(h) $I \cap \mathscr{P}(A_s \setminus \{T(s)\})$ is $\{A_{s^\frown \langle n \rangle} : n \in \omega\}$-generated.

Let see first how such a construction implies the lemma. By definition of T and conditions (a) and (b), T is 1-1. By condition (g) F_s is not empty, hence by (c) and (d) $\bigcap_{n \in \omega} F_{f|n}$ contains exactly one element for every $f \in \omega^\omega$. But then by (f) this element is equal to $\varphi^{-1}(\{T(f|n) : n \in \omega\})$, and hence $\{T(f|n) : n \in \omega\} \in I$, which proves (i).

We prove (ii) by induction on n. Assume that $B \in J$ and $B \subset \hat{s}_0 \cup \cdots \cup \hat{s}_k$ for $\hat{s}_0, \ldots, \hat{s}_k \in \omega^n$. Then in particular $\{t \in B : \mathrm{lh}\, t < n\}$ is finite. Let $B' = B \setminus \{t \in B : \mathrm{lh}\, t < n\}$. Then, by definition of T, $T(B' \cap \hat{s}_i) \in \mathscr{P}(A_{s_i} \setminus \{T(s_i)\}) \cap I$. Hence by (h) there is k_i such that $T(B' \cap \hat{s}_i) \subset A_{s_i^\frown <0>} \cup \cdots \cup A_{s_i^\frown <k_i>}$. It is easy to see that $\hat{t} \subset T^{-1}(A_t)$ for every $t \in \omega^{<\omega}$, so $B' \subset \bigcup_{i \leq k} \{\hat{t} : t = s_i^\frown < m >$ for $m \leq k_i\}$, and we are done.

The construction: Assume we already constructed F_s and A_s. Then by (g), $\varphi(F_s)$ satisfies requirements of Lemma 1. Let $\{A_n : n \in \omega\}$ be as in Lemma 1, and put $A_{s^\frown <n>} = (A_s \cap A_n) \setminus \{T(s)\}$, $F'_{s^\frown <n>} = F_s \cap \varphi^{-1}(\mathscr{F}_{s^\frown <n>})$. Then all conditions are satisfied, except perhaps (d). To make (d) true, split $F'_{s^\frown <n>}$ into finitely many closed pieces of sufficiently small diameter. It is easy to see that at least one piece must satisfy (g), and choose $F_{s^\frown <n>}$ to be such a piece. $\square$

We assume that the reader knows the concept of rank of well-founded subsets of $\omega^{<\omega}$. The key to the next lemma is the following classical characterization of analytic families of well-founded sets given by Lusin (see [K, Mo]).

Theorem 3 (Lusin). *If a family $\mathscr{B}$ of well-founded subsets of $\omega^{<\omega}$ is Σ_1^1, then $\{\mathrm{rank}(B) : B \in \mathscr{B}\}$ is bounded in ω_1.*

Lemma 3. *If I an ideal on ω is Σ_1^1, bisequential and not countably I-generated then $\hat{I} \notin \Sigma_1^1$.*

Proof. Let J be as in Lemma 2. Then by Lemma 2(i) every element of $\hat{J}$ is well founded. We prove that for every $s \in \omega^{<\omega}$ the set $\{\mathrm{rank}(B) : B \in \hat{J} \cap \mathscr{P}(\hat{s})\}$ is unbounded in ω_1. Assume that for every $s \in \omega^{<\omega}$ and every $\beta < \alpha < \omega_1$, we have $B_{s,\beta} \in \hat{J} \cap \mathscr{P}(\hat{s})$ with $\mathrm{rank}(B_{s,\beta}) \geq \beta$. Let $\{\beta_n : n \in \omega\}$ be an enumeration of α. Then by Lemma 2(ii) $B = \bigcup_{n \in \omega} B_{s^\frown <n>, \beta_n} \cup \{s\} \in \hat{J}$ and $\mathrm{rank}(B) \geq \alpha$. Hence, by the Lusin theorem $\hat{J} \notin \Sigma_1^1$, but $\hat{J} = T^{-1}(\hat{I})$, so $\hat{I} \notin \Sigma_1^1$. $\square$

3. Results

In this section we will derive Theorem 2 from the lemmas, and give some remarks.

Proof of Theorem 2. The implication from (i) to (ii) is easy. Indeed assume that $\{U_n : n \in \omega\}$ is a base of neighborhoods of x for Y. Then

$$\{(y_i) \in Y^\omega : \lim_{i \to \infty} y_i = x\} = \bigcap_{n \in \omega} \bigcup_{k \in \omega} \bigcap_{i \geq k} \{(y_i) \in Y^\omega : y_i \in U_n\}.$$

Since $\{(y_i) \in Y^\omega : y_i \in U_n\}$ is closed and open in Y^ω, the set $\{(y_i) \in Y^\omega : \lim_{i \to \infty} y_i = x\}$ is $\mathbf{F}_{\sigma\delta}$.

The implication from (ii) to (iii) is trivial.

To prove the implication from (iii) to (i), assume that $X \subset \mathbf{R}^{\omega^\omega}$, $x \in X$, Y is a countable dense set in a neighborhood of x, and Y has no countable base at x. Then $\{U \cap Y : U$ is an open neighborhood of $x\}$ generates a filter

F on Y. Since Y is countable, we can assume that F is a filter on ω. Let $I = F^*$; since Y has no countable base at x, I is not countably I-generated. Moreover, by definition of the topology in $\mathbf{R}^{\omega^\omega}$,

$$A \in F \Leftrightarrow \exists f_0, \ldots, f_n \in \omega^\omega \exists a_0, \ldots, a_n, b_0, \ldots, b_n \in \mathbf{R} \forall i \le n,$$
$$a_i < x(f_i) < b_i \wedge \forall y \in Y(\forall i \le n \, a_i < y(f_i) < b_i \Rightarrow y \in A).$$

Since X is Rosenthal, every element of X has a Borel definition. Because Y is countable the above is a Σ_1^1 definition of F. It is easy to see that, since X is bisequential, the ideal I is bisequential too. Hence I satisfies the assumptions of Lemma 3.

Assume, for a contradiction that $\{(y_i) \in Y^\omega : \lim_{i \to \infty} y_i = x\} \in \Sigma_1^1$. Then $\{B \subset Y : \forall A \in F \, B \subset^* A\} \in \Sigma_1^1$, but this set is equal to $\hat{I} = \{B \subset Y : \forall A \in I \, B \cap A =^* \emptyset\}$ and by Lemma 3, $\hat{I} \notin \Sigma_1^1$, a contradiction. □

In fact, in the above proof of Theorem 2 we make use of a condition weaker then bisequentiality of X; all that we needed was a similar condition for one special ultrafilter in Lemma 2. On the other hand, to keep the last arguments correct, every element of X has to be analytic. We do not know whether there exists any compactum satisfying all that, but not a Rosenthal one.

Lemma 2 gives us more information. To make this precise consider two ideals on $\omega^{<\omega}$, $J_0 = \{A : \exists f_0, \ldots, f_n \in \omega^\omega \, A \subset \{f_i|k : i \le n, k \in \omega\}\}$ and $J_1 = \{A : \exists f \in \omega^\omega A \subset \{s \in \omega^{<\omega} : \forall n < \text{lh } s \, s(n) < f(n)\}\}$. In fact Lemma 2 says that $J_0 \subset J \subset J_1$. Both ideals are Σ_1^1 and not countably generated, but J_0 is bisequential while J_1 is not. Nevertheless all the above, can be applied to J_1 (and its subideals) as well. J_1 is the maximal ideal satisfying the conclusion of Lemma 2. The minimal cardinality of a base of the ideal J_1 is usually denoted by $\mathbf{d}$. It is well known that both $\mathbf{d} = \omega_1 < 2^{\aleph_0}$ and $\omega_1 < 2^{\aleph_0} = \mathbf{d}$ are consistent with usual axioms of set theory. Moreover $\text{cov}(\mathscr{K}) = \{\min |\mathscr{F}| : \bigcup \mathscr{F} = \mathbf{R}$ and every element of $\mathscr{F}$ is meager$\} \le \mathbf{d}$, and $\text{cov}(\mathscr{K}) < \mathbf{d}$ is consistent (see [Mi]). It was known that a Rosenthal compactum has at a given point either a countable base or no base of cardinality smaller than $\text{cov}(\mathscr{K})$ (see [P2]). By Lemma 2 one can prove the following slightly stronger result.

Theorem 4. *Let* $X \subset \mathbf{R}^{\omega^\omega}$ *be a Rosenthal compactum,* $x \in X$ *and* $Y \subset X$ *countable. Then either* Y *has a countable base at* x, *or* Y *has no base at* x *of size less than* $\mathbf{d}$. □

The interesting open question is whether in the above $\mathbf{d}$ can be replaced by $2^{\aleph_0}$ (see [P2]).

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UNIVERSITY OF WARSAW, INSTITUT MATEMATYKI, BANACHA 2, 00-913 WARSZAWA 59, POLAND