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A NOTE ON COHERENT RINGS OF DIMENSION TWO

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ABSTRACT. It is shown that a commutative coherent domain cannot have Ng dimension 2 and a commutative coherent regular ring of Ng dimension 2 cannot have finite indecomposable decomposition.

INTRODUCTION

Throughout, R will denote a commutative ring with $1 \neq 0$. In [4, 3], H. K. Ng first introduced the finitely presented dimensions of modules and commutative rings to investigate coherent rings. Some interesting results were obtained. In order to avoid confusion we rename this dimension Ng dimension (see [1, p. 221]).

A ring R is called an (a, b, c)-ring if $\mathrm{w.gl.dim}\, R = a$, $\mathrm{gl.dim}\, R = b$, and $\mathrm{Ng\,dim}\, R = c$. it was asserted in [3] that a coherent domain of global dimension 2 must be $(1, 2, 3)$-, $(2, 2, 0)$-, $(2, 2, 2)$- or $(2, 2, 3)$-ring. In this paper, we show that a coherent domain cannot have Ng dimension 2, and hence there is no $(2, 2, 2)$-coherent domain. It is also shown that a coherent regular ring of Ng dimension 2 cannot have finite indecomposable decomposition.

Lemma. *Let R be a commutative ring. The following are equivalent for R-modules*:

(1) *Projective ideals are finitely generated.*
(2) *A projective module with finitely-generated localizations is finitely generated.* (*See* [5, *Theorem* 2.1].)

Theorem. *A coherent domain cannot have Ng dimension* 2.

Proof. Let R be a coherent domain. If $\mathrm{Ng\,dim}\, R = 2$, then there is an ideal I with $\mathrm{Ng\,dim}\, I = 1$. Hence there exists an exact sequence $0 \to K \to F \to I \to 0$ such that F is a nonfinitely-generated projective R-module and K is a finitely-generated R-module. By [4, Proposition 1.7] and [3, Proposition 2.4], R_P is a Noetherian ring for every prime ideal P of R, and hence I_P is finitely generated. Notice that K_P is also finitely generated. It follows that F_P is finitely generated for every prime ideal P. Since R is a domain, every projective ideal is invertible and hence is finitely generated. By the lemma, F is finitely generated, a contradiction. Therefore $\mathrm{Ng\,dim}\, R \neq 2$.

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Corollary 2. *Let R be a domain of global dimension* 2. *Then R must be a* $(1, 2, 3)$-, $(2, 2, 0)$- *or* $(2, 2, 3)$-*ring. Moreover, a* $(1, 2, 3)$-*domain is a Prüfer domain, a* $(2, 2, 0)$-*domain is a Noetherian domain, and a* $(2, 2, 3)$-*domain is an integrally closed coherent domain.*

Proof. First of all, R is a coherent ring. Indeed, suppose I is a finitely-generated ideal of R. There exists an exact sequence $0 \to K \to F \to I \to 0$ where F is finitely generated projective and K is projective. For any prime ideal P of R, K_P is finitely generated since R_P is coherent. By the lemma, K is finitely generated. It follows that R is coherent. On the other hand, R_P is a GCD domain by [7, Theorem 2.2], and hence R is an integrally closed domain by [4, Theorem 50]. The proof can be completed from Theorem 1, [4, Theorem 3.4], and [2, Theorem 64].

Remark. $(2, 2, 3)$-domains do exist (see [6, p. 383]).

A ring R is called a regular ring if every finitely-generated ideal of R has finite projective dimension.

Theorem 3. *Let R be a coherent regular ring. If R has Ng dimension* 2, *then R cannot have a finite indecomposable decomposition.*

Proof. Let P denote an arbitrary prime ideal of R. By [1, Corollary 6.2.4], R_P is a domain. aR_P is a flat ideal of R_P for every nonzero element a of R, and hence aR is flat. Since aR is finitely presented, aR is projective, which shows that the exact sequence $0 \to \operatorname{ann}(a) \to R \to aR \to 0$ splits. Hence there exists an ideal I such that $R = \operatorname{ann}(a) \oplus I$. If R is indecomposable, then it follows that $\operatorname{ann}(a) = 0$; that is, R is a domain, and $\operatorname{Ng}\dim R = 2$. By Theorem 1, this is a contradiction. Hence R cannot be indecomposable. Now assume that R has a finite indecomposable decomposition, $R = R_1 \oplus R_2 \oplus \cdots \oplus R_n$, say. By [4, Theorem 2.11], there is an R_i such that $\operatorname{Ng}\dim R_i = \operatorname{Ng}\dim R = 2$. On the other hand, R_i is an indecomposable coherent regular ring. From the above argument, $\operatorname{Ng}\dim R_i \neq 2$, a contradiction.

Corollary 4. *Let R be a coherent regular ring and let* $\operatorname{Ng}\dim R = 2$. *Suppose e is a nonzero idempotent of R. Then e is a finite sum of pairwise orthogonal primitive idempotents if and only if eR is a Noetherian ring.*

In particular, e is a primitive idempotent if and only if eR is a Noetherian domain.

Proof. Suppose $e = e_1 + \cdots + e_n$, where every e_i is primitive idempotent and $e_ie_j = 0$ for $i \neq j$. Then $eR = e_1R \oplus \cdots \oplus e_nR$ is a finite direct sum of indecomposable rings. On the other hand, eR is a coherent regular ring and $\operatorname{Ng}\dim eR \leq 2$. By Theorem 3, $\operatorname{Ng}\dim eR = 0$. Therefore eR is Noetherian.

The converse is clear.

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