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A NOTE ON A THEOREM OF J. DIESTEL AND B. FAIRES

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Dedicated to Professor M. Valdivia on the occasion of his sixtieth birthday

ABSTRACT. Applying a property concerning certain coverings of $l_0^\infty(X, \mathscr{A})$ that always contain some elements that are barrelled and dense in $l_0^\infty(X, \mathscr{A})$, we generalize a localization theorem of M. Valdivia, relative to vector bounded finitely additive measures (Theorem 1), and obtain two different generalizations of a theorem of J. Diestel and B. Faires ensuring that certain finitely additive measures are countably additive (Theorems 2 and 3).

The original proof of the quoted theorem of Diestel and Faires uses a theorem of Rosenthal that is not required in our proof of Theorem 3. This avoids imposing over the Valdivia's Λ_r-spaces defining the measure range space, the condition that they do not contain a copy of l^∞.

INTRODUCTION

From now onwards the word "space" will mean "locally convex Hausdorff space over the field K of the real or complex numbers." We set $\mathscr{A}$ to denote a σ-algebra of subsets of a set X and represent by $e(A)$ the characteristic function of the subset A of X. Let $l_0^\infty(X, \mathscr{A})$ be the linear space generated by the family $\{e(A), A \in \mathscr{A}\}$ endowed with the topology defined by the supremum norm. As usual, we shall identify the space $\mathbf{B}(\mathscr{A})$ of the bounded finitely additive scalar measures on $\mathscr{A}$ with the topological dual of the space $l_0^\infty(X, \mathscr{A})$, and the subspace of the countably additive scalar measures will be denoted by $\mathbf{M}(\mathscr{A})$.

A space E is dual locally complete [6] if $E'(\sigma(E', E))$ is locally complete. A space E is Γ_r [8] (Λ_r, [6]) if given any quasi-complete (locally complete) subspace G of $E^*(\sigma(E^*, E))$ such that G meets E' in a dense subspace of $E'(\sigma(E', E))$, G contains E'. B_r-complete spaces are Γ_r, and reflexive Banach spaces and Fréchet-Schwartz spaces provide some simple examples of Λ_r-spaces. For simplicity we introduce the following definition.

Definition. Given any positive integer p, a countable family of subspaces $W = \{L_{m_1 m_2 \ldots m_s},\ m_r \in \mathbb{N},\ 1 \le r \le s \le p\}$ of a linear space L is a p-net in

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L if the sequence $\{L_{m_1}, m_1 \in \mathbb{N}\}$ is increasing and covers L and for each $s \in \{2, \ldots, p\}$, $\{L_{m_1 \ldots m_{s-1} m_s}, m_s \in \mathbb{N}\}$ is increasing and covers $L_{m_1 m_2 \ldots m_{s-1}}$.

We shall denote by W_p the family $\{L_{m_1 m_2 \ldots m_p}, m_i \in \mathbb{N}, 1 \leq i \leq p\}$. In [3, Theorem 1] we have shown that if W is a p-net in $l_0^\infty(X, \mathscr{A})$, then there exists some $L_{m_1 m_2 \ldots m_p}$ that is a dense and barrelled subspace of $l_0^\infty(X, \mathscr{A})$. This result for $p = 1$ has been obtained by M. Valdivia in [7, Theorem 1] showing that $l_0^\infty(X, \mathscr{A})$ is suprabarrelled.

From the suprabarrelledness of $l_0^\infty(X, \mathscr{A})$, the following two results have been derived.

(a) Let μ be a bounded additive measure from a σ-algebra $\mathscr{A}$ on X into a space E. Let $\{F_n, n = 1, 2, \ldots\}$ be an increasing sequence of Γ_r-spaces covering a space F. If $f: E \to F$ is a linear mapping with closed graph, there is a positive integer q such that $f\mu$ is a F_q-valued bounded finite additive measure on $\mathscr{A}$ [7, Theorem 4].

(b) Let μ be a finitely additive measure on $\mathscr{A}$ with values in E, and let H be a $\sigma(E', E)$ total subset of E' such that $u\mu$ is a countably additive measure for each $u \in H$. If E is a countable inductive limit of B_r-complete spaces that do not contain l^∞, then μ is a countable additive measure [5, 9.4, p. 367].

This last result extends a well-known theorem of J. Diestel and B. Faires [1, Theorem 1.1].

Our previously quoted result of [3] enables us to generalize results (a) and (b) in Theorems 1 and 2 below. Besides, Theorem 2 has suggested to us a new generalization of the Diestel-Faires theorem, avoiding the condition that the range spaces do not contain a copy of l^∞.

Results

Theorem 1. *Let μ be a bounded additive measure on $\mathscr{A}$ with values in a space E. Suppose that F is a space with a p-net W such that each $L \in W_p$ has a locally convex topology $\mathscr{T}_L$ stronger than that induced by F, under which $L(\mathscr{T}_L)$ is a Γ_r-space. If f is a linear mapping from E into F with closed graph, then there exists a $G \in W_p$ such that $f\mu$ is a $G(\mathscr{T}_G)$-valued bounded finitely additive measure.*

Proof. As μ is bounded, the mapping $S: l_0^\infty(X, \mathscr{A}) \to E$, such that $S(e(A)) = \mu(A)$ for every $A \in \mathscr{A}$, is continuous, and therefore the linear map $T = fS$ has closed graph. By [3, Theorem 1], there is some $G \in W_p$ such that $H = T^{-1}(G)$ is dense in $l_0^\infty(X, \mathscr{A})$ and barrelled.

According to Theorems 1 and 14 of [8], the restriction of T to H admits a continuous extension U in $l_0^\infty(X, \mathscr{A})$ with values in G. As T has closed graph, $T = U$.

Theorem 2. *Let μ be a finitely additive measure on $\mathscr{A}$ with values in a space E, and let H be a $\sigma(E', E)$-total subset of E'. Suppose that E has a p-net W such that in each $L \in W_p$ there exists a locally convex topology $\mathscr{T}_L$ finer than that induced by E, under which $L(\mathscr{T}_L)$ is a sequentially complete Γ_r-space not containing any copy of l^∞. If $u\mu$ is a countably additive measure for each $u \in H$, then there exists a $G \in W_p$ such that μ is a $G(\mathscr{T}_G)$-valued countably additive vector measure.*

Proof. Let F denote the linear hull of H. The mapping S from $l_0^\infty(X, \mathscr{A})$ into E, such that $S(e(A)) = \mu(A)$ for every $A \in \mathscr{A}$, has closed graph, since by hypothesis $u\mu \in \mathbf{M}(\mathscr{A})$ for every $u \in F$. By Theorem 1 of [3], there is some $G \in W_p$ such that $K = S^{-1}(G)$ is dense in $l_0^\infty(X, \mathscr{A})$ and barrelled.

Again by Theorems 1 and 14 of [8], the restriction of S to K admits a continuous extension U in $l^\infty(X, \mathscr{A})$ with values in $G(\mathscr{T}_G)$, so μ is strongly additive because of Rosenthal's theorem [2]. Now μ is countably additive, since if $\{A_n, n = 1, 2, \dots\}$ is a sequence of pairwise disjoint subsets of X belonging to $\mathscr{A}$, we have that $\mu(\bigcup\{A_n, n = 1, 2, \dots\})$ is the only adherent point of the sequence $\{\sum(\mu(A_p), p = 1, 2, \dots, n), n = 1, 2, \dots\}$. In fact, $u\mu(\bigcup\{A_n, n = 1, 2, \dots\}) = (\sum(u\mu(A_p), p = 1, 2, \dots)$, for each $u \in H$.

In the last theorem, the subspaces L belonging to $\dot{W}_p$ do not contain a copy of l^∞ and are Γ_r with a topology stronger than the induced one. Now we are going to prove that the former theorem can also be established if these subspaces L, provided with a topology stronger than the initial one, are of the class Λ_r defined by Valdivia in [6].

We shall need the following well-known result of measure theory.

(c) Let $\{\lambda_n, n = 1, 2, \dots\}$ be a sequence of elements of $\mathbf{M}(\mathscr{A})$. If $\lim \lambda_n(A) = \lambda(A)$ for each $A \in \mathscr{A}$, then $\lambda \in \mathbf{M}(\mathscr{A})$.

This result states that $\mathbf{M}(\mathscr{A})(\sigma(\mathbf{M}(\mathscr{A}), l_0^\infty(X, \mathscr{A})))$ is sequentially complete. The next proposition shows that $\mathbf{M}(\mathscr{A})(\sigma(\mathbf{M}(\mathscr{A}), E))$ is also sequentially complete when E is a dense and barrelled subspace of $l_0^\infty(X, \mathscr{A})$. If $\mathscr{A}$ is infinite, there are in $\mathbf{M}(\mathscr{A})(\sigma(\mathbf{M}(\mathscr{A}), l_0^\infty(X, \mathscr{A})))$ bounded sequences without adherent point in $\mathbf{M}(\mathscr{A})$. In fact, let $\{A_n, n \in \mathbb{N}\}$ be a sequence of nonempty pairwise disjoint elements of $\mathscr{A}$. Let t_n be a point of A_n, and let δ_n be the Dirac measure on t_n. If $\lambda \in \mathbf{M}(\mathscr{A})$, we can find a p such that if $M\colon = \bigcup\{A_n, n \geq p\}$, then $|\lambda(M)| < 1/2$, and therefore, for $n \geq p$ we have that $|(\delta_n - \lambda)| \geq \delta_n(M) - |\lambda(M)| > 1/2$. In [4], is shown that in $l^1(\sigma(l^1, l^\infty))$ there are bounded sequences without any adherent point.

Proposition 1. *If E is a dense and barrelled subspace of $l_0^\infty(X, \mathscr{A})$ then the space $\mathbf{M}(\mathscr{A})(\sigma(\mathbf{M}(\mathscr{A}), E))$ is sequentially complete and $l_0^\infty(X, \mathscr{A})$ is contained in the bounded closure of E with respect to the dual pair $(E, \mathbf{M}(\mathscr{A}))$.*

Proof. The E-bounded subsets of $\mathbf{M}(\mathscr{A})$ are E-equicontinuous, and since E is dense in $l_0^\infty(X, \mathscr{A})$, they are also $l_0^\infty(X, \mathscr{A})$-equicontinuous. Hence the E-bounded subsets of $\mathbf{M}(\mathscr{A})$ are $l_0^\infty(X, \mathscr{A})$-bounded. Thus, the second affirmation follows.

Now let D be an absolutely convex subset of $\mathbf{M}(\mathscr{A})$ that is bounded and closed under $\sigma(\mathbf{M}(\mathscr{A}), E)$. Since D is $\sigma(\mathbf{M}(\mathscr{A}), l_0^\infty(X, \mathscr{A}))$-compact, we have that the topologies coincide in D and also the translations invariant uniformities induced by $\sigma(\mathbf{M}(\mathscr{A}), l_0^\infty(X, \mathscr{A}))$ and $\sigma(\mathbf{M}(\mathscr{A}), E)$.

Hence any $\sigma(\mathbf{M}(\mathscr{A}), E)$-Cauchy sequence $\{\lambda_n, n = 1, 2, \dots\}$ in $\mathbf{M}(\mathscr{A})$ is also $\sigma(\mathbf{M}(\mathscr{A}), l_0^\infty(X, \mathscr{A}))$-Cauchy, and by result (c) there is some $\lambda \in \mathbf{M}(\mathscr{A})$ such that $\lim \lambda_n = \lambda$ under $\sigma(\mathbf{M}(\mathscr{A}), l_0^\infty(X, \mathscr{A}))$.

In particular, when $X = N$ and $\mathscr{A} = 2^N$, we have that if E is any dense and barrelled subspace of l_0^∞ then $l^1(\sigma(l^1, E))$ is weakly sequentially complete.

Proposition 2. *Suppose that W is a p-net contained in a space F, and let f be a linear mapping from $l_0^\infty(X, \mathscr{A})$ into F having closed graph in the product*

$l_0^\infty(X, \mathscr{A})(\sigma(l_0^\infty(X, \mathscr{A}), \mathbf{M}(\mathscr{A}))) \times F$. *If each* $L \in W_p$ *has a locally convex topology* $\mathscr{T}_L$ *stronger than the induced by* F *such that* $L(\mathscr{T}_L)$ *is a* Λ_r*-space, then there is some* $G \in W_p$ *containing the range of* f *such that* f *is weakly continuous with respect to the dual pairs* $(l_0^\infty(X, \mathscr{A}), M(\mathscr{A}))$ *and* $(G, G(\mathscr{T}_G)')$.

Proof. By Theorem 1 of [3] and Proposition 1, there is some G such that $E := f^{-1}(G)$ is dual locally complete with respect to the dual pair $(E, \mathbf{M}(\mathscr{A}))$, and its bounded closure $\check{E}$ contains $l_0^\infty(X, \mathscr{A})$.

Let g be the restriction of f to E. As g has closed graph in the product $E(\sigma(E, \mathbf{M}(\mathscr{A}))) \times G(\mathscr{T}_G)$, then by [6, Theorems 2 and 6] the mapping g has a continuous extension h from $l_0^\infty(X, \mathscr{A})(\sigma(l_0^\infty(X, \mathscr{A}), \mathbf{M}(\mathscr{A})))$ with values in $G(\sigma(G, G(\mathscr{T}_G)'))$.

The mappings f and g are continuous, taking in F a locally convex topology weaker than the initial one, and both coincide in E. This fact concludes the proof.

Theorem 3. *Let* μ *be a countably additive measure on* $\mathscr{A}$ *with values in a space* E. *Suppose that* F *is a space with a p-net* W *such that each* $L \in W_p$ *has a locally convex topology* $\mathscr{T}_L$ *stronger than the induced by* F *under which* $L(\mathscr{T}_L)$ *is a* Λ_r*-space. Suppose finally that* f *is a linear mapping from* E *into* F *with closed graph. Then there exists a* $G \in W_p$ *such that* $f\mu$ *is a* $G(\mathscr{T}_G)$*-valued countably additive measure.*

Proof. The mapping $S: l_0^\infty(X, \mathscr{A})(\sigma(l_0^\infty(X, \mathscr{A}), \mathbf{M}(\mathscr{A}))) \to E(\sigma(E, E'))$, such that $S(e(A)) = \mu(A)$ for every $A \in \mathscr{A}$ is continuous, since $u\mu \in \mathbf{M}(\mathscr{A})$ for every $u \in E'$.

Then $T = fS$ has closed graph in $l_0^\infty(X, \mathscr{A})(\sigma(l_0^\infty(X, \mathscr{A}), \mathbf{M}(\mathscr{A}))) \times F$.

By Proposition 2, there is some $G \in W_p$ such that $T(l_0^\infty(X, \mathscr{A})) \subset G$ and $T: l_0^\infty(X, \mathscr{A})(\sigma(l_0^\infty(X, \mathscr{A}), \mathbf{M}(\mathscr{A}))) \to G(\sigma(G, G(\mathscr{T}_g)'))$ is continuous.

If $v \in G(\mathscr{T}_G)'$, then the continuity of T implies that $vf\mu \in \mathbf{M}(\mathscr{A})$, and consequently the Orlicz-Pettis theorem implies that $f\mu$ is $\mathscr{T}_G$-countably additive.

Corollary. *Let* μ *be an additive measure on* $\mathscr{A}$ *with values in a space* F, *and let* H *be a* $\sigma(F', F)$ *total subset of* F'. *Suppose that* F *has a p-net* W *such that each* $L \in W_p$ *has a locally convex topology* $\mathscr{T}_L$ *stronger than that induced by* F, *under which* $L(\mathscr{T}_L)$ *is a* Λ_r*-space. We also suppose that* $u\mu$ *is countably additive for every* $u \in H$. *Then there exists a* $G \in W_p$ *such that* μ *is a* $G(\mathscr{T}_G)$*-valued countably additive measure.*

Proof. The corollary follows directly from Theorem 3 and Orlicz-Pettis theorem taking $E = F(\sigma(F, \langle H \rangle))$ and f the identity map on E.

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