PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 115, Number 4, AUGUST 1992

PSEUDOCONTRACTION AND HOMOTOPY OF THE $\sin(1/x)$ CURVE

HIDEFUMI KATSUURA

(Communicated by James E. West)

ABSTRACT. We will prove that the space $\sin 1/x$ curve is not pseudocontractible using itself as the parameter space and that it has finitely many different homotopy equivalent classes of maps.

INTRODUCTION

W. Kuperburg of Auburn University has shown that a disk with a spiral around it (see Figure 1 on next page) is pseudocontractible relative to itself, while, obviously, it is not contractible. This result has not been published but is widely known among continuum theorists. Consequently, Kuperburg asked if the $\sin 1/x$ curve is also pseudocontractible relative to any continuum in Problem 28 of [2, p. 5]. The purpose of this paper is to prove that $\sin 1/x$ curve is not pseudocontractible relative to itself. As a corollary, a theorem stating the homotopy relation on the $\sin 1/x$ curve will also be given.

At the Chico Topology Conference in 1989, Charles Hagopian of Sacramento State University brought this problem to the attention of the author at the problem session. The author had the opportunity to talk with Professor Hagopian on several occasions and would like to express his sincere thanks for giving many helpful comments and thoroughly reading the manuscript.

Definitions and notations. A *continuum* is a nondegenerate compact connected metric space. A *map* or *mapping* is always a continuous function.

Definition. Let S be a continuum. Two mappings $f, g: S \to S$ are *homotopic* if there exists a map $H: S \times [0, 1] \to S$ such that $H(x, 0) = f(x)$ and $H(x, 1) - g(x)$ for every $x \in S$. S is said to be *contractible* if the identity map on S is homotopic to a constant map.

Definition. Let S and T be continua. Two maps $f, g: S \to S$ are *pseudohomotopic relative to* T if for some points $p, q \in T$, there exists a map $H: S \times T \to S$ such that $H(x, p) = f(x)$ and $H(x, q) = g(x)$ for every point $x \in S$. S is *pseudocontractible relative to* T if the identity map on S is pseudohomotopic to a constant map relative to T.

Received by the editors June 19, 1990 and, in revised form, January 21, 1991.
1991 *Mathematics Subject Classification*. Primary 55P99, 54F15.
Key words and phrases. Homotopy, contraction, pseudocontraction, uniform continuity.

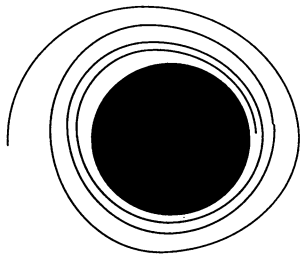

FIGURE 1. A disk with a spiral around it.

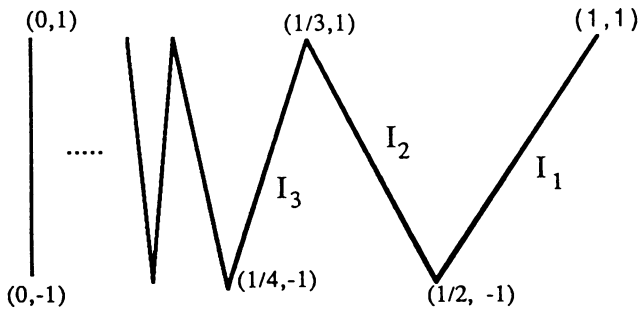

FIGURE 2. The $\sin 1/x$ curve.

Notation. Let $I_n = \{(\frac{1}{n}, y) | -1 \leq y \leq 1\}$ for every $n = 1, 2, 3, \ldots$ and $I_0 = \{(0, y) | -1 \leq y \leq 1\}$. Let $Y = I_0 \cup I_1 \cup I_2 \cup I_3 \cup \cdots$ with the metric ρ defined by $\rho[(r_1, y_1), (r_2, y_2)] = \max\{|r_1 - r_2|, |y_1 - y_2|\}$. For every $k = 1, 2, 3, \ldots$, we identify $(\frac{1}{2k-1}, -1)$ with $(\frac{1}{2k}, -1)$ and $(\frac{1}{2k}, 1)$ with $(\frac{1}{2k+1}, 1)$ in Y, and call the resulting space $\sin 1/x$ *curve* and denote it by X (see Figure 2). Let $g: Y \to X$ be the quotient map. As long as there is no confusion, if $(x, y) \in Y$ we say $(x, y) \in X$ instead of $g(x, y) \in X$, and for every $n = 0, 1, 2, \ldots$ we say $I_n \subset X$ instead of $g(I_n) \subset X$. Hence, we have $(\frac{1}{2k-1}, -1) = (\frac{1}{2k}, -1)$ and $(\frac{1}{2k}, 1) = (\frac{1}{2k+1}, 1)$ for every $k = 1, 2, 3, \ldots$. Those points $(1, 1), (\frac{1}{2}, -1), (\frac{1}{3}, 1), \ldots$ in X obtained by identification of points in Y are called *vertices* of X. For each $k = 1, 2, \ldots$ we say that the vertex $(\frac{1}{2k}, 1)$ is the *next vertex* of $(\frac{1}{2k-1}, 1)$ and $(\frac{1}{2k+1}, -1)$ is the *next vertex* of $(\frac{1}{2k}, 1)$. The metric d on X is defined by

$$d[(r_1, y_1), (r_2, y_2)] = \begin{cases} 0 & \text{if } (r_1, y_1) \text{ and } (r_2, y_2) \text{ are the same vertex}, \\ \rho((r_1, y_1), (r_2, y_2)) & \text{otherwise}. \end{cases}$$

If $x \in X$ then $N_\varepsilon(x)$ denotes the ε-neighborhood of x in X and $M_\varepsilon(x)$ denotes the component of x in $\mathrm{Cl}[N_\varepsilon(x)]$. Thus if $x \in X - I_0$ then $M_\varepsilon(x)$ is a closed neighborhood of x in X.

If $x \neq y \in X - I_0$, and if y separates $X - I_0$ between x and $(1, 1)$ or $y = (1, 1)$, we say $x > y$ and $[y, x]$ denote the set $\{z \in X : y \leq z \leq x\}$. If A, B are subsets of $X - I_0$, and if $x \in A$ and $y \in B$ implies $x < y$, then we write $A < B$.

Let n be a positive integer, and let $A \subset I_n \cup I_{n+1} \subset X$. If $a \in A$ then we define

$$a^* = \begin{cases} (\frac{1}{n}, s) & \text{if } a = (\frac{1}{n+1}, s), \\ (\frac{1}{n+1}, s) & \text{if } a = (\frac{1}{n}, s). \end{cases}$$

We also define $A^* = \{a^* : a \in A\}$. Hence, if $a, b \in A$ such that $a < b$, then $a^*, b^* \in A^*$ and $b^* < a^*$. Note that this operation "*" depends on the choice $I_n \cup I_{n+1}$. For example, if we consider $I_n \subset I_n \cup I_{n+1}$, then $I_n^* = I_{n+1}$, while if we consider $I_n \subset I_{n-1} \cup I_n$ then $I_n^* = I_{n-1}$.

Pseudocontraction

Let d denote the metric on $X \times X$ defined by

$$d[(x_1, y_1), (x_2, y_2)] = \max\{d(x_1, y_1), d(x_2, y_2)\}.$$

The following technical lemma is the heart of this paper.

Lemma 1. *Let $p_0 = (0, t) \in I_0 \subset X$, where $-1 \le t < 1$. For every $n = 1, 2, 3, \ldots$ let $p_n = (\frac{1}{n}, t)$. Let $f: X \times X \to X$ be a continuous function such that $f(x, p_0) = x$ for every $x \in X$. Choose ε such that $\frac{1}{10} > \varepsilon > 0$. Let $\delta > 0$ such that $d[f(x_1, y_1), f(x_2, y_2)] < \varepsilon$ whenever $(x_1, y_1), (x_2, y_2) \in X \times X$ and $d[(x_1, y_1), (x_2, y_2)] < \delta$. (There is such δ because f is uniformly continuous.) Let $N > \frac{2}{\delta}$ be an even integer. Let $i \ge N$ also be a fixed even integer. For every $n = 1, 2, 3, \ldots$ let $m(n, i) = m(n)$ be an integer such that $f((\frac{1}{n}, 0), p_i) \in I_{m(n)}$.*

Let $t = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_N = 1$ be a partition of the interval $[t, 1] \subset [-1, 1]$ such that $t_k - t_{k-1} < \delta$ for each $k = 1, 2, 3, \ldots, N$. (We can do this because diam$([-1, 1]) = 2$ *and $N > 2/\delta$.) Let $p_0 = x_0 = (0, t_0)$, $x_1 = (0, t_1), \ldots, x_N = (0, t_N)$, $x_{N+1} = (0, t_{N-1}), \ldots, x_{2N} = (0, t_0)$. Let $p_i = y_0 = (\frac{1}{i}, t_0) < y_1 = (\frac{1}{i}, t_1) < \cdots < y_N = (\frac{1}{i}, t_N) < y_{N+1} = (\frac{1}{i+1}, t_{N-1}) < \cdots < y_{2N} = (\frac{1}{i+1}, t_0) = p_{i+1}$.*

Conclusion. For every $k = 0, 1, 2, \ldots, N, N+1, \ldots, 2N$ and for every integer $n \ge N + 2k$, there exists a subcontinuum $A(n, k) = [a(n, k), b(n, k)] \subset I_n \cup I_{n+1} \subset X$ such that

(1) $(A(n, k) \cup A(n, k)^*) < (A(n+1, k) \cup A(n+1, k)^*)$;

(2) $f(A(n, k) \times \{x_k\}) = \begin{cases} M_{3\varepsilon}(\frac{1}{n}, 1) & \text{if } n \text{ is even}, \\ M_{3\varepsilon}(\frac{1}{n}, -1) & \text{if } n \text{ is odd}; \end{cases}$

(3) $f(a(n, k), x_k) = \begin{cases} (\frac{1}{n}, 1 - 3\varepsilon) & \text{if } n \text{ is even}, \\ (\frac{1}{n}, -1 + 3\varepsilon) & \text{if } n \text{ is odd}; \end{cases}$

$f(b(n, k), x_k) = \begin{cases} (\frac{1}{n+1}, 1 - 3\varepsilon) & \text{if } n \text{ is even}, \\ (\frac{1}{n+1}, -1 + 3\varepsilon) & \text{if } n \text{ is odd}; \end{cases}$

(4) $f(A(n, k)^* \times \{x_k\}) \subset \begin{cases} M_{4\varepsilon}(\frac{1}{n}, 1) & \text{if } n \text{ is even}, \\ M_{4\varepsilon}(\frac{1}{n}, -1) & \text{if } n \text{ is odd}; \end{cases}$

(5) $f(A(n, k) \times \{y_k\}) \subset \begin{cases} M_{4\varepsilon}(\frac{1}{m(n)}, 1) & \text{if } n \text{ is even}, \\ M_{4\varepsilon}(\frac{1}{m(n)}, -1) & \text{if } n \text{ is odd}; \end{cases}$

$$(6)\quad f(A(n,k)^* \times \{y_k\}) \subset \begin{cases} M_{5\varepsilon}(\frac{1}{m(n)}, 1) & \textit{if } n \textit{ is even}, \\ M_{5\varepsilon}(\frac{1}{m(n)}, -1) & \textit{if } n \textit{ is odd}. \end{cases}$$

Proof. If $k = 0$, then for every $n \geq N$, let

$$a(n,k) = \begin{cases} (\frac{1}{n}, 1 - 3\varepsilon) & \text{if } n \text{ is even}, \\ (\frac{1}{n}, -1 + 3\varepsilon) & \text{if } n \text{ is odd}; \end{cases}$$

$$b(n,k) = \begin{cases} (\frac{1}{n+1}, 1 - 3\varepsilon) & \text{if } n \text{ is even}, \\ (\frac{1}{n+1}, -1 + 3\varepsilon) & \text{if } n \text{ is odd}. \end{cases}$$

Then $A(n,k) = [a(n,k), b(n,k)]$ and $A(n,k) = A(n,k)^*$ for every $n \geq N$, and the conditions (1)–(6) are satisfied.

Let $k < 2N$ be a fixed integer. Suppose that the subcontinuum $A(n,k) = [a(n,k), b(n,k)]$ that satisfies the conditions (1)–(6) exists for every $n \geq N + 2k$. We will prove the existence of $A(n, k+1)$ that satisfies the conditions (1)–(6) for every $n \geq N + 2(k+1)$. To begin with, let $n = N + 2(k+1)$ for simplicity, and in Step 1 we will show how to construct $A(n, k+1)$. Also, for convenience, let $A(n-1, k+1) = A(n-1, k) \subset I_{n-1} \cup I_n$. Then

$$[A(n-1, k+1) \cup A(n-1, k+1)^*] < [A(n,k) \cup A(n,k)^*].$$

Step 1. The construction of $A(n, k+1)$. Let

$$\alpha = \max(A(n-1, k+1) \cup A(n-1, k+1)^*).$$

Then $\alpha < b(n,k)$ since $[A(n-1, k+1) \cup A(n-1, k+1)^*] < A(n,k)$ and $f(\alpha, x_k) \in M_{4\varepsilon}(\frac{1}{n}, -1)$. Thus $f(\alpha, x_{k+1}) \in M_{5\varepsilon}(\frac{1}{n}, -1)$. Since $f(b(n,k), x_k) = (\frac{1}{n+1}, 1-3\varepsilon)$, we have $f(b(n,k), x_{k+1}) \in M_{\varepsilon}(\frac{1}{n+1}, 1-3\varepsilon)$. Thus there exists a point $a \in [\alpha, b(n,k)] \subset I_n \cup I_{n+1}$ such that $f(a, x_{k+1}) = (\frac{1}{n}, 1 - 3\varepsilon)$ by the intermediate value theorem. Let $a(n, k+1) = \max\{a \in [\alpha, b(n,k)] : f(a, x_{k+1}) = (\frac{1}{n}, 1-3\varepsilon)\}$. Then $[a(n, k+1), b(n,k)] \subset I_n \cup I_{n+1}$ and $f(a(n, k+1), x_{k+1}) = (\frac{1}{n}, 1 - 3\varepsilon)$ and

$$(\#)\qquad f([a(n, k+1), b(n,k)] \times \{x_{k+1}\}) \subset M_{4\varepsilon}\left(\frac{1}{n}, 1\right).$$

Next, let $\beta = \min(A(n+1, k) \cup A(n+1, k)^*)$. Then $a(n, k+1) < b(n,k) < \beta$ and $f(\beta, x_k) \in M_{4\varepsilon}(\frac{1}{n+1}, -1)$. Hence, $f(\beta, x_{k+1}) \in M_{5\varepsilon}(\frac{1}{n+1}, -1)$. Since we also have that $f(a(n, k+1), x_{k+1}) = (\frac{1}{n}, 1 - 3\varepsilon)$, there exists $b \in [a(n, k+1), \beta] \subset I_n \cup I_{n+1}$ such that $f(b, x_{k+1}) = (\frac{1}{n+1}, 1 - 3\varepsilon)$ by the intermediate value theorem. Let $b(n, k+1) = \min\{b \in [a(n, k+1), \beta] : f(b, x_{k+1}) = (\frac{1}{n+1}, 1 - 3\varepsilon)\}$. Let $A(n, k+1) = [a(n, k+1), b(n, k+1)]$. Then $A(n, k+1) \subset I_n \cup I_{n+1}$, $f(a(n, k+1), x_{k+1}) = (\frac{1}{n}, 1 - 3\varepsilon)$,

$$f(b(n, k+1), x_{k+1}) = (\frac{1}{n+1}, 1 - 3\varepsilon),$$

and $f(A(n, k+1) \times \{x_{k+1}\}) = M_{3\varepsilon}(\frac{1}{n}, 1)$.

Claim 1. $f(A(n, k+1)^* \times \{x_{k+1}\}) \subset M_{4\varepsilon}(\frac{1}{n}, 1)$.

First, we will show that $f(a(n, k+1)^*, x_{k+1}) \in I_n \cup I_{n+1}$, where $A(n, k+1)^* = [b(n, k+1)^*, a(n, k+1)^*]$. Suppose this is not the case.

Then $f(a(n, k+1)^*, x_{k+1}) \in I_{n+2} \cup I_{n+3} \cup \cdots$ or $f(a(n, k+1)^*, x_{k+1}) \in I_1 \cup I_2 \cup \cdots \cup I_{n-1}$.

Case 1.1. Assume that $f(a(n, k+1)^*, x_{k+1}) \in I_{n+2} \cup I_{n+3} \cup \cdots$. Consider $[a(n, k+1), b(n, k)] \subset I_n \cup I_{n+1}$. We have $f(b(n, k)^*, x_k) \in M_{4\varepsilon}(\frac{1}{n}, 1)$ by property (4). Hence, $f(b(n, k)^*, x_{k+1}) \in M_{5\varepsilon}(\frac{1}{n}, 1)$. Also, since $f(a(n, k+1), x_{k+1}) = (\frac{1}{n}, 1-3\varepsilon)$, we must have $f(a(n, k+1)^*, x_{k+1}) \in N_\varepsilon(\frac{1}{n}, 1-3\varepsilon)$. Since $a(n, k+1) < b(n, k)$, we have $b(n, k)^* < a(n, k+1)^*$. Thus there exists a point

$$c \in [b(n, k)^*, a(n, k+1)^*] = [a(n, k+1), b(n, k)]^* \subset I_n \cup I_{n+1}$$

such that $f(c, x_{k+1}) = (\frac{1}{n+1}, -1)$ by the intermediate value theorem. But then $c^* \in [a(n, k+1), b(n, k)]$ and $f(c^*, x_{k+1}) \in N_\varepsilon(\frac{1}{n+1}, -1)$. This is a contradiction to (#).

Case 1.2. Assume that $f(a(n, k+1)^*, x_{k+1}) \in I_1 \cup I_2 \cup \cdots \cup I_{n-1}$. Then as in Case 1.1, there exists a point

$$c \in [b(n, k)^*, a(n, k+1)^*] = [a(n, k+1), b(n, k)]^* \subset I_n \cup I_{n+1}$$

such that $f(c, x_{k+1}) = (\frac{1}{n}, -1)$. But then $c^* \in [a(n, k+1), b(n, k)]$ and $f(c^*, x_{k+1}) \in N_\varepsilon(\frac{1}{n}, -1)$. Again, this is a contradiction to (#).

Therefore, we have proved that $f(a(n, k+1)^*, x_{k+1}) \in I_n \cup I_{n+1}$. But since $f(A(n, k+1)^* \times \{x_{k+1}\})$ is a continuum, and since for every $z \in A(n, k+1)$ we have $d[f(z^*, x_{k+1}), f(z, x_{k+1})] < \varepsilon$, we must have

$$f(A(n, k+1)^* \times \{x_{k+1}\}) \subset [I_n \cup I_{n+1}] \cap N_{4\varepsilon}(\frac{1}{n}, 1)$$

by the uniform continuity of f. Hence, we have $f(A(n, k+1)^* \times \{x_{k+1}\}) \subset M_{4\varepsilon}(\frac{1}{n}, 1)$.

Claim 2. $f(A(n, k+1) \times \{y_{k+1}\}) \subset M_{4\varepsilon}(\frac{1}{m(n)}, 1)$.

First, we will show that $f(a(n, k+1), y_{k+1}) \in M_{4\varepsilon}(\frac{1}{m(n)}, 1)$. Suppose $f(a(n, k+1), y_{k+1}) \notin M_{4\varepsilon}(\frac{1}{m(n)}, 1)$. Since $f(a(n, k+1), x_{k+1}) = (\frac{1}{n}, 1-3\varepsilon)$ and since $d(x_{k+1}, y_{k+1}) < \delta$, we must have $f(a(n, k+1), y_{k+1}) \in N_{4\varepsilon}(\frac{1}{m(n)}, 1)$ by the uniform continuity of f. By the inductive hypothesis, we have $f(b(n, k), y_k) \in M_{4\varepsilon}(\frac{1}{m(n)}, 1)$. Hence by the uniform continuity of f again, we have $f(b(n, k), y_{k+1}) \in M_{5\varepsilon}(\frac{1}{m(n)}, 1)$. This implies that there must exist a point $c \in [a(n, k+1), b(n, k)]$ such that $f(c, y_{k+1}) = (\frac{1}{m(n)}, -1)$ or $f(c, y_{k+1}) = (\frac{1}{m(n)+1}, -1)$ by the intermediate value theorem so that $f(c, y_{k+1}) \in N_\varepsilon(\frac{1}{m(n)}, -1)$. This implies that $f(c, x_{k+1}) \in N_{2\varepsilon}(\frac{1}{m(n)}, -1)$. This is a contradiction to (#). Therefore, we have proved that $f(a(n, k+1), y_{k+1}) \in M_{4\varepsilon}(\frac{1}{m(n)}, 1)$. But since $f(A(n, k+1) \times \{x_{k+1}\}) = M_{3\varepsilon}(\frac{1}{n}, 1)$ and $d(x_{k+1}, y_{k+1}) < \delta$, we have $f(A(n, k+1) \times \{y_{k+1}\}) \subset M_{4\varepsilon}(\frac{1}{m(n)}, 1)$.

Claim 3. $f(A(n, k+1)^* \times \{y_{k+1}\}) \subset M_{5\varepsilon}(\frac{1}{m(n)}, 1)$.

The proof of this is similar to the proof of Claim 2. First, we will show that $f(a(n, k+1)^*, y_{k+1}) \in M_{5\varepsilon}(\frac{1}{m(n)}, 1)$. Suppose $f(a(n, k+1)^*, y_{k+1}) \notin M_{5\varepsilon}(\frac{1}{m(n)}, 1)$. Since $f(a(n, k+1)^*, x_{k+1}) \in M_\varepsilon(\frac{1}{n}, 1-3\varepsilon)$ and since

$$d[(a(n, k+1)^*, x_{k+1}), (a(n, k+1)^*, y_{k+1})] < \delta,$$

we must have $f(a(n, k+1)^*, y_{k+1}) \in N_{4\varepsilon}(\frac{1}{m(n)}, 1)$ by the uniform continuity of f. By the inductive hypothesis, we have $f(b(n, k)^*, y_k) \in M_{5\varepsilon}(\frac{1}{m(n)}, 1)$. Hence by the uniform continuity of f again, we have $f(b(n, k)^*, y_{k+1}) \in M_{6\varepsilon}(\frac{1}{m(n)}, 1)$. This implies that there must exist a point $c \in [b(n, k)^*, a(n, k+1)^*]$ such that $f(c, y_{k+1}) \in N_\varepsilon(\frac{1}{m(n)}, -1)$ by the intermediate value theorem as in the proof of Claim 2. This implies that $f(c^*, x_{k+1}) \in N_{2\varepsilon}(\frac{1}{m(n)}, -1)$ and $c^* \in [a(n, k+1), b(n, k)]$. This is a contradiction to (#). Hence, we have $f(A(n, k+1)^* \times \{y_{k+1}\}) \subset M_{5\varepsilon}(\frac{1}{m(n)}, 1)$.

Therefore, we have constructed $A(n, k+1)$ that satisfies conditions (2)–(6) for k being replaced by $k+1$.

Claim 4. Moreover, we have $[A(n, k+1) \cup A(n, k+1)^*] < [A(n+1, k) \cup A(n+1, k)^*]$.

Since $b(n, k+1) < \beta$, we have $A(n, k+1) < [A(n+1, k) \cup A(n+1, k)^*]$. So we will show that $A(n, k+1)^* < [A(n+1, k) \cup A(n+1, k)^*]$.

Since $\beta = \min[A(n+1, k) \cup A(n+1, k)^*]$, we have that $\beta \in I_{n+1}$. Hence, we consider β as an element of $I_n \cup I_{n+1}$ so that $\beta^* \in I_n$. By the inductive hypothesis, we have $\beta > b(n, k)^*$ so that $\beta^* < b(n, k)$. Moreover, since $f(\beta, x_{k+1}) \in M_{5\varepsilon}(\frac{1}{n+1}, -1)$, we have $f(\beta^*, x_{k+1}) \in M_{6\varepsilon}(\frac{1}{n+1}, -1)$ by the continuity of f.

Suppose $\beta^* \geq a(n, k+1)$. Then we have that $\beta^* \in [b(n, k), a(n, k+1)]$. By (#), we have $f(\beta^*, x_{k+1}) \in M_{4\varepsilon}(\frac{1}{n}, 1)$. But this is impossible since $d[(\frac{1}{n}, 1), (\frac{1}{n+1}, -1)] = 2$, so that $M_{6\varepsilon}(\frac{1}{n+1}, -1) \cap M_{4\varepsilon}(\frac{1}{n}, 1) = \varnothing$. Therefore, we have $\beta^* < a(n, k+1)$ so that $\beta = (\beta^*)^* > a(n, k+1)^*$. This proves that

$$A(n, k+1)^* < [A(n+1, k) \cup A(n+1, k)^*].$$

Step 2. The construction of $A(n+1, k+1)$.
This is very similar to the construction of $A(n, k+1)$. More precisely, in Step 1, replace n by $n+1$, and $1-3\varepsilon$ by $-1+3\varepsilon$ and replace each vertex by its next vertex. Then $A(n+1, k+1)$ satisfies not only conditions (2)–(6) but also (1) in the statement of the lemma for k being replaced by $k+1$ since the choice for α in this case will be $\alpha = \max(A(n, k+1) \cup A(n, k+1)^*)$.

By continuing this process, we can construct $A(n+2, k+1)$, $A(n+3, k+1)$, ... Therefore, this proves that the lemma holds for every $k = 0, 1, 2, \ldots, N, N+1, \ldots, 2N$ by the mathematical induction.

Lemma 2. *The* $\sin 1/x$ *curve is not contractible.*

Proof. If X is contractible, then X must be arcwise connected. But this is a contradiction.

The main theorem is as follows.

Theorem 1. *The* $\sin 1/x$ *curve* X *is not pseudocontractible relative to itself.*

Proof. On the contrary, suppose there exists a map $f: X \times X \to X$ such that for some $p, q, r \in X$, $f(x, p) = x$ and $f(x, q) = r$ for every $x \in X$. Then the points p and q cannot be on the same arc-component of X since X is not contractible.

Case 1. Suppose $q \in I_0$ and $p \in X - I_0$. If $y \in X - I_0$ then $f(X \times \{y\})$ is a nondegenerate subcontinuum of X that contains points of $X - I_0$ and I_0; otherwise it would imply that X is contractible. This implies that $I_0 \subset f(X \times \{y\})$ for every $y \in X - I_0$. Because of the continuity of f, we must have $I_0 \subset f(X \times \{y\})$ for every $y \in I_0$. This is a contradiction to $f(X \times \{q\}) = \{r\}$.

Case 2. Suppose $p \in I_0$ and $q \in X - I_0$. Let $p = (0, t)$, $-1 \leq t \leq 1$ and $p_n = (\frac{1}{n}, t)$ for every $n = 1, 2, 3, \ldots$. Let $\frac{1}{10} > \varepsilon > 0$. Let $\delta > 0$ such that $d[f(x_1, y_1), f(x_2, y_2)] < \varepsilon$ whenever $(x_1, y_1), (x_2, y_2) \in X \times X$ and $d[(x_1, y_1), (x_2, y_2)] < \delta$. Let $N > \frac{2}{\delta}$ be an even integer. For every $n = 1, 2, 3, \ldots$ and for every integer $i \geq N$, let $m(n, i)$ be an integer such that $f((\frac{1}{n}, 0), p_i) \in I_{m(n,i)}$. Since $q \in X - I_0$, we have that $f(X \times \{y\}) \subset X - I_0$ for every $y \in X - I_0$. Hence, $f(X \times \{p_n\}) \subset X - I_0$ for every $n = 1, 2, \ldots$. But $p_n \to p$. So there exists an integer $i \geq N$ such that for some $n_0 \geq 5N + 2$, we have $m(n_0, i) < m(n_0, i + 1)$.

Suppose i is an even integer. The case when i is an odd integer is similar. If $t = 1$ then we must have $m(n_0, i) = m(n_0, i + 1)$ since i is even, which is impossible. So $t < 1$. Let $t = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_N = 1$ be a partition of the interval $[t, 1] \subset [-1, 1]$ such that $t_k - t_{k-1} < \delta$ for each $k = 1, 2, 3, \ldots, N$. Let $p_i = y_0 = (\frac{1}{i}, t_0) < y_2 = (\frac{1}{i}, t_2) < \cdots < y_N = (\frac{1}{i}, t_N) = (\frac{1}{i+1}, t_N) < y_{N+1} = (\frac{1}{i+1}, t_{N-1}) < y_{N+2} = (\frac{1}{i+1}, t_{N-2}) < \cdots < y_{2N} = (\frac{1}{i+1}, t_0) = p_{i+1}$. Then $\{y_0 < y_2 < \cdots < y_N < y_{N+1} < y_{N+2} < \cdots < y_{2N}\}$ is a partition of $[p_i, p_{i+1}]$ such that $d(y_k, y_{k-1}) < \delta$. Then by Lemma 1, we know that for every $k = 0, 1, 2, \ldots, N, N+1, \ldots, 2N$ and for every integer $n \geq N + 2k$, there exists a subcontinuum $A(n, k) \subset I_n \cup I_{n+1}$ such that

$$f([A(n, k) \cup A(n, k)^*] \times \{y_k\}) \subset \begin{cases} M_{5\varepsilon}(\frac{1}{m(n,i)}, 1) & \text{if } n \text{ is even}, \\ M_{5\varepsilon}(\frac{1}{m(n,i)}, -1) & \text{if } n \text{ is odd}. \end{cases}$$

In particular, since $y_{2N} = p_{i+1}$, we have for every integer $n \geq N + 2(2N) = 5N$ there exists a subcontinuum $A(n, 2N) \subset I_n \cup I_{n+1}$ such that

$$f([A(n, 2N) \cup A(n, 2N)^*] \times \{p_{i+1}\}) \subset \begin{cases} M_{5\varepsilon}(\frac{1}{m(n,i)}, 1) & \text{if } n \text{ is even}, \\ M_{5\varepsilon}(\frac{1}{m(n,i)}, -1) & \text{if } n \text{ is odd}. \end{cases}$$

Suppose n_0 is even. Then $(\frac{1}{m(n_0-1,i)}, -1) = (\frac{1}{m(n_0,i)}, -1)$. Hence,

$$f([A(n_0, 2N) \cup A(n_0, 2N)^*] \times \{p_{i+1}\}) \subset M_{5\varepsilon}\left(\frac{1}{m(n_0, i)}, 1\right)$$

and

$$f([A(n_0 - 1, 2N) \cup A(n_0 - 1, 2N)^*] \times \{p_{i+1}\}) \subset M_{5\varepsilon}\left(\frac{1}{m(n_0, i)}, -1\right).$$

Moreover, $[A(n_0, 2N) \cup A(n_0, 2N)^*] \cap I_{n_0} \neq \varnothing$ and $[A(n_0 - 1, 2N) \cup A(n_0 - 1, 2N)^*] \cap I_{n_0} \neq \varnothing$. Thus there exists points $\xi, \psi \in I_{n_0}$ such that

$$f(\xi, p_{i+1}) \in M_{5\varepsilon}\left(\frac{1}{m(n_0, i)}, 1\right) \quad \text{and} \quad f(\psi, p_{i+1}) \in M_{5\varepsilon}\left(\frac{1}{m(n_0, i)}, -1\right).$$

On the other hand, since $d(p, p_{i+1}) < \delta$, we must have

$$f(I_{n_0} \times \{p_{i+1}\}) \subset I_{m(n_0, i+1)} \cup M_\varepsilon\left(\frac{1}{m(n_0, i+1)}, 1\right) \cup M_\varepsilon\left(\frac{1}{m(n_0, i+1)}, -1\right).$$

Hence, $m(n_0, i) < m(n_0, i+1)$ implies that

$$f(\xi, p_{i+1}) \notin f(I_{n_0} \times \{p_{i+1}\}) \quad \text{or} \quad f(\psi, p_{i+1}) \notin f(I_{n_0} \times \{p_{i+1}\}).$$

This is a contradiction to $\xi, \psi \in I_{n_0}$. We can obtain a similar contradiction if n_0 is an odd integer. This proves the theorem.

Theorem 2. *Let Y be a nondegenerate indecomposable continuum such that each of which composants is arcwise connected. Let Z be a continuum that has a proper nondegenerate arc component. Then Z is not pseudocontractible relative to Y.*

Proof. Let $p \in Y$ and $f: Z \times Y \to Z$ a continuous function such that $f(z, p) = z$ for each $z \in Z$. Let K be the composant of p in Y. Let I be a nondegenerate arc component in Z. For every $y \in K$, $I \subset f(Z \times \{y\})$ since there is an arc in K from p to y. But then this implies that for every $y \in Y$, $I \subset f(Z \times \{y\})$ since K is dense in Y. Therefore, Z is not pseudocontractible relative to Y.

Corollary 2.1. *Let Y be a nondegenerate indecomposable continuum such that each of which composants is arcwise connected. Then the* $\sin 1/x$ *curve is not pseudocontractible relative to Y.*

Corollary 2.2. *Let Y be a nondegenerate indecomposable continuum such that each of which composants is arcwise connected. Then Y is not pseudocontractible relative to itself.*

Proof. Let $p \in Y$ and $f: Y \times Y \to Y$ a continuous function such that $f(x, p) = x$ for each $x \in Y$. Let K be the composant of p in Y. For every $y \in K$, $f(Y \times \{y\}) = Y$ since K is arcwise connected. But then this implies that for every $x \in Y$, $f(Y \times \{x\}) = Y$ since K is dense in Y. Therefore, Y is not pseudocontractible relative to itself.

Note. Knaster's bucket handles and solenoids are examples of indecomposable continua with the property that each composant is arcwise connected.

Questions. (1) Is the $\sin 1/x$ curve pseudocontractible relative to the pseudoarc?

(2) [W. Kuperburg] Is the pseudoarc pseudocontractible relative to the pseudoarc? (See Problem 118 [1, p. 385].)

Homotopy

Lemma 3. *Let $f: X \times [0, 1] \to X$ be a continuous function such that $f(x, 0) = x$ for every $x \in X$. Choose $\frac{1}{10} > \varepsilon > 0$. Let $\delta > 0$ such that $d[f(x_1, y_1), f(x_2, y_2)] < \varepsilon$ whenever $(x_1, y_1), (x_2, y_2) \in X \times [0, 1]$ and $d[(x_1, y_1), (x_2, y_2)] < \delta$. (There is such δ because f is uniformly continuous.) Let $N > \frac{1}{\delta}$ be an even integer. Let $t_0 = 0, t_1 = \frac{1}{N}, t_2 = \frac{2}{N}, \ldots, t_k =$*

$\frac{k}{N}, \dots, t_N = 1$ *be a partition of the interval* $[0, 1]$. *Then for every* $k = 0, 1, 2, \dots, N$ *and for every integer* $n \geq N + 2k$, *there exists a subcontinuum* $A(n, k) = [a(n, k), b(n, k)] \subset I_n \cup I_{n+1} \subset X$ *such that*

(1) $(A(n, k) \cup A(n, k)^*) < (A(n+1, k) \cup A(n+1, k)^*)$;

(2) $f(A(n, k) \times \{t_k\}) = \begin{cases} M_3(\frac{1}{n}, 1) & \text{if } n \text{ is even}, \\ M_3(\frac{1}{n}, -1) & \text{if } n \text{ is odd}; \end{cases}$

(3) $f(a(n, k), t_k) = \begin{cases} (\frac{1}{n}, 1 - 3\varepsilon) & \text{if } n \text{ is even}, \\ (\frac{1}{n}, -1 + 3\varepsilon) & \text{if } n \text{ is odd}; \end{cases}$

$f(b(n, k), t_k) = \begin{cases} (\frac{1}{n+1}, 1 - 3\varepsilon) & \text{if } n \text{ is even}, \\ (\frac{1}{n+1}, -1 + 3\varepsilon) & \text{if } n \text{ is odd}; \end{cases}$

(4) $f(A(n, k)^* \times \{t_k\}) \subset \begin{cases} M_{4\varepsilon}(\frac{1}{n}, 1) & \text{if } n \text{ is even}, \\ M_{4\varepsilon}(\frac{1}{n}, -1) & \text{if } n \text{ is odd}. \end{cases}$

Proof. This is an easy consequence of Lemma 1.

Theorem 3. *Let* $f: X \times [0, 1] \to X$ *be a continuous function such that* $f(x, 0) = x$ *for every* $x \in X$. *Choose* $\frac{1}{10} > \varepsilon > 0$. *Then there exists an integer* M *such that for every* $n \geq M$ *and for every* $t \in [0, t]$, *we have* $f(I_n \times \{t\}) \cap M_{2\varepsilon}(\frac{1}{n}, 1) \neq \varnothing$ *and* $f(I_n \times \{t\}) \cap M_{2\varepsilon}(\frac{1}{n}, -1) \neq \varnothing$.

Proof. Let us use the same notations as in Lemma 3. Let $t \in [0, 1]$ and let k be an integer such that $t_k \leq t \leq t_{k+1}$. Let $M = N + 2N + 1 = 3N + 1$. Let $n \geq M$. Then by Lemma 3, there exist $\xi \in I_{n-1} \cup I_n$ and $\psi \in I_n \cup I_{n+1}$ such that $f(\xi, t_k) = (\frac{1}{n}, -1)$ and $f(\psi, t_k) = (\frac{1}{n}, 1)$. Let $x \in \{\xi, \xi^*\} \cap I_n$ and $y \in \{\psi, \psi^*\} \cap I_n$. Then $f(x, t_k) \in M_\varepsilon(\frac{1}{n}, -1)$ and $f(y, t_k) \in M_\varepsilon(\frac{1}{n}, 1)$. By the uniform continuity of f, we have $f(x, t) \in M_{2\varepsilon}(\frac{1}{n}, -1)$ and $f(y, t) \in M_{2\varepsilon}(\frac{1}{n}, 1)$. Therefore, we have shown that $f(I_n \times \{t\}) \cap M_{2\varepsilon}(\frac{1}{n}, -1) \neq \varnothing$ and $f(I_n \times \{t\}) \cap M_{2\varepsilon}(\frac{1}{n}, -1) \neq \varnothing$.

The next theorem implies that the identity map on X is not homotopic to any other maps on X constructed by shifting period on the identity map.

Theorem 4. *For every integer* $n = \dots, -2, -1, 0, 1, 2, \dots$, *we define a continuous function* $h_n: X \to X$ *by the following rule*:

Let $y \in [-1, 1]$ *and* k *a positive integer. Then*

$$h_n(0, y) = \begin{cases} (0, y) & \text{if } n \text{ is even}, \\ (0, -y) & \text{if } n \text{ is odd} \end{cases}$$

and

$$h_n(1/k, y) = \begin{cases} (\frac{1}{k+n}, y) & \text{if } n = 0, 2, 4, \dots, \\ (\frac{1}{k+n}, -y) & \text{if } n = 1, 3, 5, \dots, \\ (1, 1) & \text{if } n \text{ is negative and } k \leq |n|, \\ (\frac{1}{k+n}, y) & \text{if } n = -2, -4, -6, \dots \text{ and } k > |n|, \\ (\frac{1}{k+n}, -y) & \text{if } n = -1, -3, -5, \dots \text{ and } k > |n|; \end{cases}$$

hence h_n *is homotopic to* h_m *if and only if* $n = m$.

Proof. Let n be an integer. Let $f_n: X \times [0, 1] \to X$ be a continuous function such that $f_n((x, y), 0) = h_n(x, y)$. Then as in the above theorem, we can

prove that there exist an integer $K > |n|$ such that for every $k \geq K$ and for every $t \in [0, 1]$, we have

$$f_n(I_n \times \{t\}) \cap M_\varepsilon\left(\frac{1}{n+k}, 1\right) \neq \varnothing \quad \text{and} \quad f_n(I_k \times \{t\}) \cap M_\varepsilon\left(\frac{1}{n+k}, -1\right) \neq \varnothing.$$

This proves that if m is an integer different from n, then h_n is not homotopic to h_m.

References

1. Wayne Lewis, *Continua theory problems*, Topology Proc. **8** (1983).
2. Houston Problem Book, Department of Mathematics, University of Houston.

Department of Mathematics, San Jose State University, San Jose, California 95192-0103