

THE FOURIER-BESSEL SERIES REPRESENTATION OF THE PSEUDO-DIFFERENTIAL OPERATOR $(-x^{-1}D)^\nu$

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ABSTRACT. For a certain Fréchet space F consisting of complex-valued C^∞ functions defined on $I = (0, \infty)$ and characterized by their asymptotic behaviour near the boundaries, we show that:

(I) The pseudo-differential operator $(-x^{-1}D)^\nu$, $\nu \in \mathbb{R}$, $D = d/dx$, is an automorphism (in the topological sense) on F ;

(II) $(-x^{-1}D)^\nu$ is almost an inverse of the Hankel transform h_ν in the sense that

$$h_\nu \circ (x^{-1}D)^\nu(\varphi) = h_0(\varphi), \quad \forall \varphi \in F, \quad \forall \nu \in \mathbb{R};$$

(III) $(-x^{-1}D)^\nu$ has a Fourier-Bessel series representation on a subspace $F_b \subset F$ and also on its dual F'_b .

1. INTRODUCTION

Let F be the space of all C^∞ complex-valued function $\varphi(x)$ defined on $I = (0, \infty)$ such that

$$(1.1) \quad \varphi(x) = \sum_{i=0}^k a_i x^{2i} + o(x^{2k})$$

near the origin and is *rapidly decreasing* as $x \rightarrow \infty$.

For $\nu > -\frac{1}{2}$, we define a ν th order Hankel transform h_ν on F by

$$(1.2) \quad \Phi(y) = [h_\nu \varphi(x)](y) = \int_0^\infty \varphi(x) \mathcal{J}_\nu(xy) dm(x),$$

where

$$dm(x) = m'(x) dx = [2^\nu \Gamma(\nu + 1)]^{-1} x^{2\nu+1} dx,$$

$$\mathcal{J}_\nu(x) = 2^\nu \Gamma(\nu + 1) x^{-\nu} J_\nu(x),$$

and $J_\nu(x)$ is the Bessel function of order ν . The inversion formula for (1.2) is given by [1, 3, 4],

$$(1.3) \quad \varphi(x) = \int_0^\infty \Phi(y) \mathcal{J}_\nu(xy) dm(y).$$

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In this paper we will show that for every real ν :

- (I) The pseudodifferential operator $(-x^{-1}D)^\nu$ is a topological automorphism on F .
- (II) The Hankel transform h_ν is also an automorphism on F .
- (III) On F , $(-x^{-1}D)^\nu$ is almost an inverse of h_ν in the sense that

$$[h_\nu \circ (-x^{-1}D)^\nu](\varphi) = h_0(\varphi), \quad \varphi \in F.$$

- (IV) On a certain subspace $F_b \subset F$ and on its dual F'_b , $(-x^{-1}D)^\nu$ has Fourier-Bessel series representations.

In the sequel all automorphisms are topological automorphisms.

2. PRELIMINARIES

For any real $\nu \neq -\frac{1}{2}$, F_ν is the space of all C^∞ complex-valued function $\varphi(x)$ defined on I such that

$$(2.1) \quad \gamma_{m,k}^\nu(\varphi) = \sup_{x \in I} |x^m \Delta_{\nu,x}^k \varphi(x)| < \infty,$$

for each $m, k = 0, 1, 2, \dots$, where

$$\Delta_{\nu,x} = D^2 + x^{-1}(2\nu + 1)D.$$

F_ν is a Fréchet space. Its topology is generated by the countable family of separating seminorms $\{\gamma_{m,k}^\nu\}_{m,k=0,1,2,\dots}$, [5; 7, p. 8].

Theorem 2.1(i) of Lee [3, p. 429] shows that $F_\nu = F_\mu = F$ (as a set) for each $\nu, \mu (\neq -\frac{1}{2}) \in \mathbb{R}$. Hence for each $\nu \neq -\frac{1}{2}$, we have a topology T_ν on F generated by the countable family of seminorms $\gamma_{m,k}^\nu$. Hence (F, T_ν) is a Fréchet space. When $\nu = -\frac{1}{2}$, $F_{-1/2} \neq F$, since the factor $x^{-1}(2\nu + 1)D$ in $\Delta_{\nu,x}$, responsible for the even nature of $\varphi(x) \in F_\nu(x)$ near the origin, vanishes. For example $e^{-x} \in F_{-1/2}$ but $e^{-x} \notin F_\nu$; $\nu \neq -\frac{1}{2}$.

Definition. Zemanian [7, 8] defined a Hankel transform \bar{h}_ν ($\nu \geq -\frac{1}{2}$) by

$$(2.2) \quad \Psi(y) = [\bar{h}_\nu \psi(x)](y) = \int_0^\infty \psi(x) \sqrt{xy} J_\nu(xy) dx.$$

He proved that \bar{h}_ν is an automorphism on the space H_ν that consists of complex-valued C^∞ functions defined on I and satisfies the relation

$$(2.3) \quad \bar{\gamma}_{m,k}^\nu(\psi) = \sup_{x \in I} |x^m (x^{-1}D)^k [x^{-\nu-1/2} \psi(x)]| < \infty,$$

for each $m, k = 0, 1, 2, \dots$, where $D = d/dx$.

The following theorem is a key result for the latter development of our theory.

Theorem 2.1. Let ν, μ be real number $\neq -\frac{1}{2}$. Then

- (I) The operation $\varphi \rightarrow x^{\nu+1/2} \varphi$ is an homeomorphism from F onto H_ν .
- (II) $(x^{-1}D)^n: F \rightarrow F$ is an automorphism on F .
- (III) (F, T_ν) and (F, T_μ) are equivalent topological spaces.
- (IV) $h_\nu(\varphi) = (-1)^n [h_{\nu+n}(x^{-1}D)^n] \varphi$, for $\varphi \in F$, $\nu \geq -\frac{1}{2}$, and $n = 0, 1, 2, \dots$.

Notation. In view of the Theorem 2.1(I, III), we will always write $x^{\nu+1/2}\varphi(x) = \bar{\varphi}(x) \in H_\nu$ for $\varphi \in F$ and drop the suffix ν from T_ν . So henceforth the topological linear space (F, T) will be denoted by F .

Proof. (I) By induction on n and noting that

$$(2.4) \quad \Delta_{\nu, x} = x^2(x^{-1}D)^2 + 2(\nu + 1)(x^{-1}D),$$

it can be proved that

$$(2.5) \quad \Delta_{\nu, x}^n = x^{2n}(x^{-1}D)^{2n} + a_1x^{2(n-1)}(x^{-1}D)^{2n-1} + \dots + a_n(x^{-1}D)^n,$$

where a_i 's are the constants depending on ν . Now $\varphi \in F$ iff $\bar{\varphi} \in H_\nu$ (follows from Remark II of Lee [3]) and taking $a_0 = 1$, it follows from (2.5) that

$$(2.6) \quad \gamma_{m, k}^\nu(\varphi) \leq \sum_{i=k}^{2k} a_{2k-i} \bar{\gamma}_{m+2(i-k), i}^\nu(\bar{\varphi}),$$

proving the continuity of the inverse operation $\bar{\varphi} \rightarrow x^{-\nu-1/2}\varphi$. Invoking the Open Mapping Theorem [6, p. 172], F being Fréchet space, we complete the proof.

(II) Let φ_j be a sequence tending to zero in F . Then $\bar{\varphi}_j \rightarrow 0$ in H_ν for arbitrary $\nu \neq -\frac{1}{2}$. Hence

$$\begin{aligned} \gamma_{m, k}^\nu[(x^{-1}D)^n \varphi_j(x)] &\leq \sum_{i=k}^{2k} a_{2k-i} \bar{\gamma}_{m+2(i-k), i+n}^\nu(\bar{\varphi}_j) \quad (\text{from (2.6)}) \\ &\rightarrow 0 \quad \text{as } j \rightarrow \infty. \end{aligned}$$

It remains to be shown that $(x^{-1}D)^n$ is bijective. It is enough to prove this for $n = 1$. So, let $x^{-1}D\varphi_1(x) = x^{-1}D\varphi_2(x)$ for $\varphi_1, \varphi_2 \in F$. Hence $\varphi_1(x) - \varphi_2(x) = \text{constant}$. But $\varphi_1(x)$ and $\varphi_2(x)$ are of rapid descent as $x \rightarrow \infty \Rightarrow \varphi_1(x) = \varphi_2(x)$. Now let $\psi(x) \in F$. Then $\varphi(x) = -\int_x^\infty t\psi(t) dt$, defined uniquely (since ψ is of rapid descent as $x \rightarrow \infty$) in F , is such that $x^{-1}D\varphi(x) = \psi(x)$. So we see that $(x^{-1}D)^n$ is a continuous bijection on F . The space F being a Fréchet space, the Open Mapping Theorem shows that $(x^{-1}D)^n$ is a bicontinuous bijection on (F, T_ν) for each $\nu \in \mathbb{R} - \{\frac{1}{2}\}$.

(III) Let $\nu = \mu + a$, $a \in \mathbb{R}$, and φ_n be a sequence tending to zero in (F, T_μ) . Then

$$\begin{aligned} \gamma_{m, k}^\nu(\varphi_n) &= \sup_{x \in I} |x^m [\Delta_{\mu, x} + 2a(x^{-1}D)]^k \varphi_n(x)| \\ &\leq \sup_{x \in I} x^m \left[\left| \sum_{i=0}^k \Delta_{\mu, x}^{k-i} (2ax^{-1}D)^i \varphi_n(x) \right| + \sum_{i=0}^k |(2ax^{-1}D)^{k-i} \Delta_{\mu, x}^i \varphi_n(x)| \right. \\ &\quad \left. + \text{terms of the type } |\Delta_{\mu, x}^{i_1} (2ax^{-1}D)^{i_2} \Delta_{\mu, x}^{i_3} \dots \varphi_n(x)| \right. \\ &\quad \left. \text{and } |(2ax^{-1}D)^{j_1} \Delta_{\mu, x}^{j_2} (2ax^{-1}D)^{j_3} \dots \varphi_n(x)| \right] \\ &\quad (\text{where } i_1 + i_2 + i_3 + \dots = j_1 + j_2 + j_3 + \dots = k) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty, \text{ for each } m, k = 0, 1, 2, \dots, \end{aligned}$$

since $\Delta_{\mu,x}^i$ and $(x^{-1}D)^i$ are continuous on (F, T_μ) .

(IV) follows from integration by parts and induction on n .

Remark 1. It can be shown that on F

$$\Delta_{\nu,x}^k \circ (x^{-1}D)^n = (x^{-1}D)^n \circ \Delta_{\nu-n,x}^k.$$

The proof follows by induction on k .

Definition. In view of Theorem 2.1(IV), we define the Hankel transform h_ν formally for any $\nu \in \mathbb{R}$, as

$$(2.7) \quad h_\nu(\varphi) = h_{\nu+n} \circ (-x^{-1}D)^n \varphi, \quad \varphi \in F,$$

where n is so chosen that $\nu + n > -\frac{1}{2}$.

This is a well-defined definition as $(x^{-1}D)^n$ is an automorphism.

Definition. Let F' be the dual space of F . Then for $f \in F'$, define the generalized Hankel transform $h_\nu f (= \hat{f})$ of f by

$$\langle h_\nu f, h_\nu \varphi \rangle = \langle f, \varphi \rangle \quad \forall \varphi \in F, \nu \in \mathbb{R}.$$

Theorem 2.2. For $\nu \in \mathbb{R}$, h_ν is an automorphism on F and hence on F' .

Proof. Let $\varphi(x) \in F$. Then

$$(2.8) \quad \begin{aligned} h_\nu(\varphi) &= \Phi(y) = \int_0^\infty (x^{-1}D)^{2n} \varphi(x) \mathcal{I}_{\nu+2n}(xy) dm(x) \\ &= y^{-\mu-1/2} \bar{h}_\mu(\bar{\psi}(x))(y), \quad \text{where } \mu = \nu + 2n > -\frac{1}{2}, \end{aligned}$$

where

$$\bar{\psi}(x) = x^{\mu+1/2} \psi(x) = x^{\mu+1/2} (x^{-1}D)^{2n} \varphi(x).$$

Let

$$\begin{aligned} \varphi_m(x) \rightarrow 0 \text{ in } F &\Rightarrow \bar{\psi}_m(x) \rightarrow 0 \text{ in } H_\mu \text{ (Theorem 2.1(I))}, \\ &\Rightarrow \bar{h}_\mu(\bar{\psi}_m) \rightarrow 0 \text{ in } H_\mu, \\ &\Rightarrow h_\nu(\varphi_m) \rightarrow 0 \text{ in } F. \end{aligned}$$

Now \bar{h}_μ , the Zemanian Hankel transform, being bijective, (2.8) shows that h_ν is a bijection. Hence use of the Open Mapping Theorem completes the proof.

Writing $\nu = 0$ in (2.7) we get

$$h_0(\varphi) = h_n \circ (-x^{-1}D)^n(\varphi), \quad \varphi \in F.$$

The above equation motivates us to propose the following

Definition. For $\nu \in \mathbb{R}$, define $(-x^{-1}D)^\nu$ by

$$(2.9) \quad (-x^{-1}D)^\nu(\varphi) = h_\nu^{-1} \circ h_0(\varphi), \quad \varphi \in F.$$

Then $(-x^{-1}D)^\nu$ is clearly an automorphism on F for each real ν . From equation (2.9) we get

$$(2.10) \quad (-x^{-1}D)^\nu \varphi(x) = \int_0^\infty dm(y) \mathcal{I}_\nu(xy) \int_0^\infty dm(x) \varphi(x) \mathcal{I}_0(xy).$$

For distributions $f \in F'$, define $(-x^{-1}D)^\nu$ by

$$(2.11) \quad \langle (-x^{-1}D)^\nu f, \varphi \rangle = \langle f, (-x^{-1}D)^\nu \varphi \rangle, \quad \varphi \in F.$$

So we modify Theorem 2.1(II) to give our main result.

Theorem 2.3. *The pseudodifferential operator $(-x^{-1}D)^\nu$ is an automorphism on F and hence on F' for each $\nu \in \mathbb{R}$.*

3. THE FOURIER-BESSEL SERIES EXPANSION OF $(-x^{-1}D)^\nu$

Equation (2.10) gives the integral representation of the operator $(-x^{-1}D)^\nu$. To get the Fourier-Bessel series expansion, we modify our leading function space F suitably as follows (similar to the ones as in Zemanian [7, 9]).

For $b > 0$, define

$$(3.1) \quad F_b = \{\varphi \in F \mid \varphi \equiv 0 \text{ for } x > b\}.$$

The topology of F_b is generated by a countable family of seminorms

$$(3.2) \quad \gamma_k^\nu(\varphi) = \sup_{0 < x < b} |\Delta_{\nu,x}^k \varphi(x)| < \infty, \quad k = 0, 1, 2, \dots$$

Clearly all the topologies obtained by choosing different ν 's are equivalent.

Remark 2. Without loss of generality, we may take $\nu > -\frac{1}{2}$.

Definition. We define finite Hankel transform h_ν by

$$(3.3) \quad \Phi(z) = [h_\nu \varphi](z) = \int_0^b \varphi(x) \mathcal{J}_\nu(xz) dm(x).$$

Then $\Phi(z)$ is an even entire function by Griffith's Theorem [2, 9]. Let $z = y + iw$ and $G_b = \{\Phi(z) \mid \Phi(z) \text{ is an even entire function satisfying (3.4)}\}$.

$$(3.4) \quad \alpha_b^k(\Phi) = \sup_{z \in \mathbb{C}} |e^{-b|w|} z^{2k} \Phi(z)| < \infty,$$

for $k = 0, 1, 2, \dots$. Then G_b is a linear topological space with α_b^k as seminorms.

Both the spaces F_b and G_b are Hausdorff, locally convex topological linear spaces satisfying the axiom of first countability. They are sequentially complete spaces.

Theorem 3.1. *h_ν is an homeomorphism from F_b onto G_b .*

Proof. Let $\varphi \in F_b$. Then

$$\Phi(z) = h_{\nu+2m}[(x^{-1}D)^{2m}\varphi(x)], \quad \text{for } m \in \mathbb{N}.$$

Hence

$$z^{2m}\Phi(z) = \int_0^b x^{2\nu+2m+1}[(x^{-1}D)^{2m}\varphi(x)](xz)^{-\nu} J_{\nu+2m}(xz) dz.$$

From the asymptotic formula

$$J_\nu(z) \sim \sqrt{2/\pi z} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \quad |z| \rightarrow \infty, \quad |\arg z| < \pi,$$

and from the fact that $z^{-\nu} J_{\nu+m}(z)$ is an entire function, it follows that for all x and z ,

$$|e^{-b|w|}(xz)^{-\nu} J_{\nu+2m}(xz)| < C_{m\nu} \quad (\text{a constant}).$$

Hence

$$(3.5) \quad \alpha_b^m(\Phi) \leq C_{m\nu} b^{2(m+\nu+1)} \gamma_0^\nu [(x^{-1}D)^{2m} \varphi(x)] < \infty.$$

$(x^{-1}D)^{2m}$ being an automorphism (also on F_b), (3.5) implies the continuity of h_ν . h_ν is clearly injective. For any $\Phi(z) \in G_b$, take

$$\varphi(x) = \int_0^\infty \Phi(y) \mathcal{J}_\nu(xy) dm(y).$$

Then it follows from Griffith's Theorem [2] that φ is zero almost everywhere for $x > b$. Also,

$$\begin{aligned} \gamma_k^\nu(\varphi) &= \sup_{0 < x < b} \left| \Delta_{\nu,x}^k \int_0^\infty \Phi(y) \mathcal{J}_\nu(xy) dm(y) \right| \\ &= \sup_{0 < x < b} \left| \int_0^\infty \Phi(y) (-1)^k y^{2\nu+2k+1} (xy)^{-\nu} J_\nu(xy) dy \right| \\ &< \infty, \quad \text{for each } k = 0, 1, 2, \dots, \end{aligned}$$

since $\Delta_{\nu,x}^k [(xy)^{-\nu} J_\nu(xy)] = (-1)^k y^{2k} (xy)^{-\nu} J_\nu(xy)$, $\Phi(y)$ is of rapid descent as $y \rightarrow \infty$, and $(xy)^{-\nu} J_\nu(xy)$ is bounded for $0 < y < \infty$. Therefore, $\varphi \in F_b$. Hence h_ν is surjective. Now the Open Mapping Theorem completes the proof.

Theorem 3.2. *Let $\varphi \in F_b$. Then*

$$(3.6) \quad \varphi(x) = \lim_{\varepsilon \rightarrow 0^+} \frac{2}{b^2} \sum_{n=1}^\infty \lambda_\varepsilon(x) \left(\frac{\lambda_n}{x}\right)^\nu \frac{J_\nu(x\lambda_n)}{J_{\nu+1}^2(b\lambda_n)} \Phi_0(\lambda_n),$$

where the λ_n 's are the positive roots of $J_\nu(bz) = 0$ arranged in the ascending order and for $0 < \varepsilon < b/4$,

$$\lambda_\varepsilon(x) = \begin{cases} E(x/2\varepsilon), & 0 < x < 2\varepsilon, \\ 1, & 2\varepsilon \leq x \leq b - 2\varepsilon, \\ 1 - E\left(\frac{x - b + 2\varepsilon}{2\varepsilon}\right), & b - 2\varepsilon < x < b, \\ 0, & x \geq b, \end{cases}$$

and $E(u) = \int_0^u \exp[1/x(x - 1)] dx / \int_0^1 \exp[1/x(x - 1)] dx$.

Proof. Trivial. See also [5].

Theorem 3.2 gives the required Fourier-Bessel Series expansion for the pseudo-differential operator $(-x^{-1}D)^\nu$, which we obtain in the following

Theorem 3.3 (The Fourier-Bessel Series). *For $\varphi \in F_b$, we have*

$$(3.7) \quad [(-x^{-1}D)^\nu] \varphi(x) = \lim_{\varepsilon \rightarrow 0^+} \frac{2}{b^2} \sum_{n=1}^\infty \lambda_\varepsilon(x) \left(\frac{\lambda_n}{x}\right)^\nu \frac{J_\nu(x\lambda_n)}{J_{\nu+1}^2(b\lambda_n)} \Phi_0(\lambda_n),$$

where $\Phi_0(y) = h_0[\varphi(x)](y)$.

Proof. Equation (2.9) along with Theorem 3.2 gives the required proof.

Note that

$$|\lambda_n^{\nu+1/2} \Phi_0(\lambda_n)| \leq A_{k\nu} \lambda_n^{(\nu+1/2)-2k},$$

$A_{k\nu}$ constants and $[J_\nu(x\lambda_n)/x^\nu \lambda_n^{1/2} J_{\nu+1}^2(b\lambda_n)]$ is smooth and bounded on $0 < x < b$, $0 < \lambda_n < \infty$.

Hence the truncation error

$$E_N = \lim_{\varepsilon \rightarrow 0^+} \frac{2}{b^2} \sum_{n=N+1}^\infty \lambda_\varepsilon(x) \left(\frac{\lambda_n}{x}\right)^\nu \frac{J_\nu(x\lambda_n)}{J_{\nu+1}^2(b\lambda_n)} \Phi_0(\lambda_n)$$

has exponential decay for large N .

The Theorem 3.3 gives the Fourier-Bessel series representation of the operator $(-x^{-1}D)^\nu$ on the testing function space F_b . We wish to investigate the nature of the Fourier-Bessel series for the pseudodifferential operator $(-x^{-1}D)^\nu$ on the distribution space F'_b .

The spaces F'_b and G'_b are dual spaces of F_b and G_b , respectively. They are assigned the weak topologies generated by the seminorms

$$P_\varphi(f) = |\langle f, \varphi \rangle|, \quad \varphi \in F_b, f \in F'_b,$$

and

$$P_\varphi(h_\nu f) = |\langle h_\nu f, h_\nu \varphi \rangle|, \quad h_\nu \varphi \in G_b, h_\nu f \in G'_b,$$

respectively.

Both the spaces are sequentially complete.

Definition. For $f \in F'_b$, $\varphi \in F_b$, we define the generalized finite Hankel transform $h_\nu f$ by

$$(3.8) \quad \langle h_\nu f, h_\nu \varphi \rangle = \langle f, \varphi \rangle.$$

Theorem 3.4. For $\nu \in \mathbb{R}$, h_ν is an homeomorphism from F'_b onto G'_b .

Theorem 3.5. For every $\varepsilon \in (0, b/4)$ and each $f \in F'_b$, the function

$$(3.9) \quad \hat{f}_\varepsilon(y) = \langle f(x), y^{-\nu-1/2} \lambda_\varepsilon(x) m'(y) \mathcal{J}_\nu(xy) \rangle,$$

where $\lambda_\varepsilon(x)$ is defined as in Theorem 3.2, is a smooth function of slow growth, and defines a regular generalized function in G'_b .

Proof. Note that $(x^{-1}D)^k \lambda_\varepsilon(x)$ is bounded on $0 < x < b$ for each k . Using (2.6), it is easy to see that $y^{-\nu-1/2} \lambda_\varepsilon(x) m'(y) \mathcal{J}_\nu(xy) \in F_b$. Hence (3.9) is well defined. The rest of the proof is similar to that of Zemanian [8, Lemma 12].

Theorem 3.6. The finite Hankel transform $h_\nu f$ of a generalized function f in F'_b is the distributional limit, as $\varepsilon \rightarrow 0^+$, of the family $\hat{f}_\varepsilon(z)$ defined by (3.9).

Proof. Trivial.

Theorem 3.7. Let $f \in F'_b$ and $\hat{f} = h_\nu f$. Then in the sense of convergence in F'_b , we have

$$(3.10) \quad f(x) = \lim_{N \rightarrow \infty} \frac{2}{b^2} \sum_{n=1}^N \frac{x^{\nu+1}}{\sqrt{\lambda_n}} [J_\nu(x\lambda_n)/J_{\nu+1}^2(b\lambda_n)] \cdot \hat{f}(\lambda_n).$$

Proof. The proof follows easily from Theorems 3.2 and 3.6.

Remark 3. For $f \in F'_b$, such that either f is regular or $\text{supp } f \subset (0, b]$, the limit of $\hat{f}_\varepsilon(z)$ as $\varepsilon \rightarrow 0^+$ exists as an ordinary function and is equivalent to the finite Hankel transform of f [5].

A consequence of the above theorem is the following

Theorem 3.8. Let $f, g \in F'_b$. If $(h_\nu f)(\lambda_n) = (h_\nu g)(\lambda_n)$, for $n = 1, 2, 3, \dots$, then $f = g$ and $h_\nu f = h_\nu g$.

Definition. For $f \in F'_b$, define $(-x^{-1}D)^\nu f$ by

$$(3.11) \quad \langle (-x^{-1}D)^\nu f, \varphi \rangle = \langle f, (-x^{-1}D)^\nu \varphi \rangle, \quad \varphi \in F_b, \nu \in \mathbb{R}.$$

From equations (2.9), (3.8), and (3.11), it follows that

$$\begin{aligned} \langle (-x^{-1}D)^\nu f, \varphi \rangle &= \langle f, (-x^{-1}D)^\nu \varphi \rangle \\ &= \langle h_0^{-1} h_\nu f, \varphi \rangle, \quad f \in F'_b, \varphi \in F_b. \end{aligned}$$

Hence

$$(3.12) \quad (-x^{-1}D)^\nu f = h_0^{-1} h_\nu f \quad \text{on } F'_b.$$

Applying Theorem 3.7 to equation (3.12) we get

Theorem 3.10 (The Fourier-Bessel Series). Let $f \in F'_b$ and $\hat{f} = h_\nu f$. Then in the sense of convergence in F'_b , we have

$$(3.13) \quad (-x^{-1}D)^\nu f(x) = \lim_{N \rightarrow \infty} \frac{2}{b^2} \sum_{n=1}^N \frac{x}{\sqrt{\lambda_n}} [J_0(x\lambda_n)/J_1^2(b\lambda_n)] \hat{f}(\lambda_n).$$

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