POLYDISCS AND NONTANGENTIAL LIMITS

KOHUR GOWRISANKARAN

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ABSTRACT. A well-known result states that for all bounded n-harmonic functions on the polydisc $\mathbb{D}^n$ the nontangential limits exist for (Lebesgue) almost every element of the n-torus. In this paper it is shown that a similar result is not in general valid for bounded quotients of two positive n-harmonic functions. Necessary and sufficient conditions on a n-harmonic function $u > 0$ are given to ensure the existence "almost everywhere" of the nontangential limits of the quotients w/u in the case (i) for all n-harmonic functions w such that w/u is bounded and in the case (ii) for all n-harmonic functions w that are ' u-quasi-bounded.'

Let $\mathbb{D}^n$ be the n-dimensional polydisc and w an n-harmonic function on $\mathbb{D}^n$, i.e., a continuous real-valued function that is harmonic separately in each variable. The following is a well-known result concerning a class of those functions [8]. Suppose w is n-harmonic and bounded on $\mathbb{D}^n$; then for Lebesgue almost every $\underline{t} \in \mathbb{T}^n$ as $\underline{z} = (z_1, \ldots, z_n)$ tends to $\underline{t} = (t_1, \ldots, t_n)$ where each $z_j \to t_j$ nontangentially (and to stress the condition independent of the other variables) $w(\underline{z})$ converges to a real number. In terms of potential theoretical expectations it is natural to expect the result to extend to bounded quotients of positive n-harmonic functions. More precisely, suppose $u > 0$ is an n-harmonic function on $\mathbb{D}^n$ that is represented as the integral of a finite Borel measure μ_u on $\mathbb{T}^n$ relative to the product of the Poisson kernels [4, 8]. Let w be any u-bounded (i.e., $|w/u|$ bounded) n-harmonic function on $\mathbb{D}^n$. It is natural to expect the nontangential limits of w/u to exist for μ_u almost every element of $\mathbb{T}^n$.

The first main result of this paper will prove this expectation to be false. We give further a necessary and sufficient condition on u under which the result is valid for all u-bounded n-harmonic functions. With a slightly different perspective it is natural to ask if the nontangential limits exist almost everywhere on $\mathbb{T}^n$ for quasi-bounded n-harmonic functions. We consider the problem in the general situation of u-quasi-bounded n-harmonic functions. We recall that a positive n-harmonic function is said to be u-quasi-bounded if it is the limit

of an increasing sequence of positive u-bounded functions. We give necessary and sufficient conditions on u to ensure the existence of nontangential limits of w/u at μ_u almost every element of $\mathbb{T}^n$ and for all u-quasi-bounded n-harmonic functions w. One of the principal tools used in our proofs is the existence of iterated fine limits of quotients of n-harmonic functions [7]. Another development, which is probably of independent interest, is the introduction of a restructured PWB (for Perron-Wiener-Brelot) problem in our situation. We want to thank the referee for some valuable comments and, in particular, for the much simpler proof of Theorem 2.

In what follows we assume that $u > 0$ is a n-harmonic function on $\mathbb{D}^n$ and is represented uniquely by the finite Borel measure μ_u on $\mathbb{T}^n$ relative to the product of the Poisson kernels.

We now have

Lemma 1. *Let $w > 0$ be a n-harmonic function represented by the finite Borel measure ν_w on $\mathbb{T}^n$. Then the iterated nontangential limits of w/u exist for μ_u almost every element of $\mathbb{T}^n$. The limit function (determined up to a set of μ_u measure zero) is a Radon-Nikodym derivative of the absolutely continuous part of ν_w relative to μ_u.*

Proof. We know that the iterated fine limits of w/u exist μ_u almost everywhere and the limit function is the required Radon-Nikodym derivative [7]. However, by [1, Theorem 4], we conclude that at every stage the existence of the fine limit implies that of the nontangential limit, equal to the fine limit. Thus the iterated nontangential limit of w/u exists μ_u almost everywhere and verifies the required condition. The proof is complete.

Corollary A. *If in addition w is u-quasi-bounded then $g\, d\mu_u$ represents the function w where g is the iterated limit function.*

Proof. It is simple to verify that $w > 0$ is u-quasi-bounded if and only if the measure ν_w on $\mathbb{T}^n$ representing w is absolutely continuous relative to μ_u. The corollary is then easily seen to be true.

The following corollary is an immediate consequence.

Corollary B. *Suppose w is as in the above corollary and assume, in addition, that the nontangential limit of w/u exists for μ_u almost every element of $\mathbb{T}^n$. Then the nontangential limit function g is such that $g\, d\mu_u$ is the representing measure of w.*

Definition. Suppose f is an extended real-valued function on $\mathbb{D}^n$. For the element $\underline{t} = (t_1, \dots, t_n) \in \mathbb{T}^n$, we refer to nontangential $\limsup f(\underline{z})$ as $\underline{z} = (z_1, \dots, z_n) \to \underline{t}$ as the extended real number a that is the supremum of the limit values of $f(z)$ when $\underline{z} \to \underline{t}$ nontangentially as explained at the beginning. It is easily verified that a is also the supremum as $\underline{\theta} \to \pi/2$ of the $\limsup f(\underline{z})$ as $\underline{z} \to \underline{t}$ and $\underline{z}$ stays in the product of Stolz domains of semivertical angles $(\theta_1, \dots, \theta_n) = \underline{\theta}$ [1].

Theorem 2. *There exist n-harmonic functions u and w both > 0, w/u bounded such that the nontangential limit of w/u does not exist for a set of elements of $\mathbb{T}^n$, of positive $(> 0)\mu_u$ measure, where μ_u is the canonical (finite Borel) measure on $\mathbb{T}^n$ representing u.*

Proof. We recall that there exists an n-harmonic function $w > 0$ such that the nontangential $\liminf w = 0$ and nontangential $\limsup w = +\infty$ for (Lebesgue) almost every element of $\mathbb{T}^n$ [2, 5]. Let $u = w + 1$ where w is the function as above. Then $u > 0$ is n-harmonic on $\mathbb{D}^n$ and w is u-bounded. Further we note that the canonical measure μ_u is such that $\mu_u \geq$ the Lebesgue measure on $\mathbb{T}^n$. Clearly, the nontangential $\liminf w/u = 0$ and nontangential $\limsup w/u = 1$ for (Lebesgue) almost every element of $\mathbb{T}^n$. In particular, the same holds for a subset of $\mathbb{T}^n$ with μ_u-measure > 0. This proves the theorem.

Theorem 3. *Let $u > 0$ be a n-harmonic function with the representing measure μ_u. For every Borel subset $E \subset \mathbb{T}^n$, let u_E be the n-harmonic function with representing measure $\chi_E \, d\mu_u$. The following two conditions are equivalent.*

(1) For every u-bounded n-harmonic function w, the nontangential limit of w/u exists for μ_u almost every element of $\mathbb{T}^n$.

(2) For every Borel set $E \subset \mathbb{T}^n$, the nontangential limit of u_E/u equals zero for μ_u almost every element in $\mathbb{T}^n \backslash E$.

Proof. Suppose that the condition (1) is verified for u. Then by the corollary B, the nontangential limit of $u_E\backslash u$ equals χ_E, μ_u almost everywhere on $\mathbb{T}^n$.

Conversely, suppose that condition (2) is verified for u. Let E be an arbitrary Borel set. Since (2) is valid both for E and $\mathbb{T}^n\backslash E$, we conclude easily that the nontangential limit of u_E/u equals χ_E, μ_u almost everywhere. Suppose f is any bounded Borel function on $\mathbb{T}^n$ and $\sigma(f)$ the n-harmonic function that is the integral of the product of Poisson kernels relative to the signed measure $f \, d\mu$. Every u-bounded n-harmonic function is necessarily of this form. Simple measure theoretic arguments let us conclude that the nontangential limit of $\sigma(f)/u$ exists μ_u almost everywhere on $\mathbb{T}^n$. Then we conclude by Corollary B that the above limit equals f, μ_u almost everywhere. The proof is complete.

Let us now prove the following restricted version of a minimum principle for n-harmonic functions.

Theorem 4. *Let w be a n-harmonic function on $\mathbb{D}^n$ such that $w/u \geq -\alpha$ for some positive number α. Let further the nontangential $\liminf$ of w/u be ≥ 0 for μ_u almost every element of $\mathbb{T}^n$. Then $w \geq 0$.*

Proof. The positive n-harmonic function $w + \alpha u$ can be uniquely expressed as the sum $w_1 + w_2$ of positive n-harmonic functions where the representing measure of w_1 (resp. w_2) is absolutely continuous (resp. singular) relative to μ_u. From the hypothesis of the theorem and Lemma 1 we know the iterated (fine as well as) nontangential limit of $(w/u + \alpha)$ exists and majorizes α, μ_u almost everywhere. We further conclude that $w_1 \geq \alpha u$.

Hence, $w + \alpha u = (w_1 - \alpha u) + \alpha u + w_2$ where each term on the right is a positive n-harmonic function. Hence $w \geq 0$ completing the proof.

We now proceed to consider the nontangential behavior of u-quasi-bounded n-harmonic functions. Let us assume that the positive n-harmonic function u verifies either of the equivalent hypothesis stated in Theorem 3. Let us have the convention that $\text{Inf} \, \phi = \infty$. For an extended real-valued function f defined

on $\mathbb{T}^n$ we define the following lower envelope:

$$S_0(f) = \mathrm{Inf} \left\{ w : \begin{array}{l} w \quad n\text{-harmonic and } u\text{-quasi-bounded}, \\ w/u \text{ lower bounded on } \mathbb{D}^n, \\ n - t. \liminf w/u \geq f, \ \mu_u \text{ a.e. on } \mathbb{T}^n. \end{array} \right.$$

Modifying the envelope introduced by Doob [3] we define for all $f \geq 0$ on $\mathbb{T}^n$

$$S_1(f) = \mathrm{Inf} \left\{ w : \begin{array}{l} w \geq 0, \ n\text{-harmonic on } \mathbb{D}^n, \\ w \text{ is } u\text{-quasi-bounded}, \\ n - t. \limsup w/u \geq f, \ \mu_u \text{ a.e. on } \mathbb{T}^n. \end{array} \right.$$

Theorem 5. $S_0(f)$ *is identically* $\pm\infty$ *or it is a* n*-harmonic function on* $\mathbb{D}^n$.

Proof. We may assume that $S_0(f) \not\equiv +\infty$. Suppose w_1 and w_2 are two of the functions in the class defining $S_0(f)$. Define for each positive integer m,

$$h_m = \text{greatest } n\text{-harmonic minorant of } \min(w_1, w_2, mu)$$

[4]. Clearly h_m exists and is u-bounded and hence the nontangential limit of h_m/u exists μ_u almost everywhere.

Now, $v_m = \min(w_1, w_2, mu) - h_m$ is a nonnegative n-superharmonic function with greatest n-harmonic minorant zero. Further the iterated nontangential as well as iterated fine limit of v_m/u exists μ_u a.e. and is hence $= 0$, μ_u a.e. It follows therefore that $n - t. \liminf v_m/u = 0$, μ_u a.e. Hence, the $n - t. \liminf \min \frac{1}{u}(w_1, w_2, mu) = n - t. \lim h_m/u$, μ_u a.e. This implies that the $n - t. \lim h_m/u$ is $\geq \min(f, m)$, μ_u a.e. Clearly the sequence of n-harmonic functions h_m increases and by Harnack property [4] converges to a n-harmonic function h as m tends to ∞ and $h \leq \min(w_1, w_2)$. Let now h' be the greatest n-harmonic minorant of $\min(w_1, w_2)$. Clearly, h'/u is lower bounded and further

$$n - t. \liminf h'/u \geq n - t. \liminf h/u \geq \min(f, m)$$

μ_u a.e. for each m. We conclude that $h' \geq S_0(f)$. Now it is easy that $S_0(f)$ may be realized as the limit of a decreasing directed family of n-harmonic functions and is hence, by Harnack property, necessarily $-\infty$ identically or a n-harmonic function on $\mathbb{D}^n$. The proof is complete.

Lemma 6. $S_0(f)$ *introduced above satisfies the following properties.*
  (a) $S_0(f)$ *increases with increasing* f.
  (b) *Suppose* $f_n \uparrow f$, $f_1 \geq 0$ *then* $S_0(f_n) \uparrow S_0(f)$.
  (c) $S_o(f) \geq -S_0(-f)$.

Proof. The first property is trivial and the third one is an immediate consequence of Theorem 4. The proof of (b) follows the familiar pattern. Suppose $f_n \geq 0$ and increases to f. We need to consider the only case when $\lim S_0(f_n) \not\equiv +\infty$ for each n. Fix $\underline{z} \in \mathbb{D}^n$ and $\varepsilon > 0$. Choose for each n, a u-quasi-bounded n-harmonic function $w_n \geq 0$, w_n in the defining family of $S_0(f_n)$ such that $S_0(f_n)(\underline{z}) + \varepsilon/2^n > w_n(\underline{z})$. It is simple to verify that for each n, $S_0(f_n)$ is u-quasi-bounded. Clearly the limit v of (the increasing sequence) $S_0(f_n)$ is n-harmonic and also u-quasi-bounded. Let us now set $w = v + \sum_1^\infty [w_n - S_0(f_n)]$. Clearly, w is a positive u-quasi-bounded

n-harmonic function such that nontangential $\liminf w/u \geq f$, μ_u a.e. Hence $w \geq S_0(f)$. It follows that $S_0(f)(\underline{z}) \leq v(\underline{z}) + \varepsilon$. The proof is completed easily.

Theorem 7. *For all Borel functions* $f \geq 0$

$$S_0(f) = \sigma(f) = \int f \mathbb{P} \, d\mu_u,$$

where $\mathbb{P}$ *is the product of the Poisson kernels.*

Proof. Consider first a Borel function f such that $0 \leq f \leq M$ for some M. $\sigma(f) \geq 0$ and besides by our hypothesis, $\sigma(f)/u$ has nontangential limits equal to f, μ_u a.e. and hence $\sigma(f) \geq S_0(f)$. However, it is easy to check that $-\sigma(f) \geq S_0(-f)$ or $\sigma(f) \leq -S_0(-f) \leq S_0(f)$. Hence, $\sigma(f) = S_0(f)$ in this case. In general, if $f \geq 0$ and Borel, set $f_n = \min(f, n)$. Then, the above lemma and monotone convergence theorem gives us

$$\sigma(f) = \lim \sigma(f_n) = \lim S_0(f_n) = S_0(f).$$

The proof is complete.

Now we introduce a stronger condition to ensure the existence of nontangential limits for u-quasi-bounded functions.

(H)   Let $f \geq 0$ be Borel and μ_u integrable on $\mathbb{T}^n$. Then the nontangential limit of $\sigma(f)/u$ exists and equals zero for μ_u almost every $\underline{t} \in N_f = \{\underline{t} : f(t) = 0\}$.

It is easy to deduce the hypothesis (2) of Theorem 3 in case u satisfies the above condition (H); hence it is stronger than the requirement of the existence of limits for u-bounded functions.

Theorem 8. *Suppose the n-harmonic function* $u > 0$ *satisfies the condition* (H) *above. Then for all* μ_u *integrable function* $f \geq 0$ *on* $\mathbb{T}^n$, $S_0(f) = S_1(f)$.

Proof. It is clear that $S_0(f) \geq S_1(f)$. We want to prove the opposite inequality. As before, let us first assume that f is in addition bounded. Let w be ≥ 0, n-harmonic and u-quasi-bounded and verify nontangential $\limsup w/u \geq f$, μ_u a.e. Suppose $w = \sigma(g)$ for some $g \geq 0$ and Borel and let $v = \min(\sigma(g), \sigma(f))$. Since the nontangential $\lim \sigma(f)/u = f$, μ_u a.e., we have that the nontangential $\limsup v/u \geq f$, μ_u a.e.

Now, it is easy to verify that $\sigma(\min(f, g)) = h$ is the greatest n-harmonic minorant of v. Further,

$$v - h = \min[\sigma(g - \min(f, g)), \sigma(f - \min(f, g))].$$

Let $E_1 = \{\underline{t} \in \mathbb{T}^n : g(\underline{t}) > f(\underline{t})\}$ and E_2 the complement of E_1. Clearly, for all $\underline{t} \in E_1$, $\min(f(\underline{t}), g(\underline{t})) = f(\underline{t})$ and hence, nontangential $\lim h/u = \min(f(\underline{t}), g(\underline{t})) = f(\underline{t})$ except on a set of μ_u measure zero. However, for μ_u almost every $\underline{t} \in E_2$ that is the set where the μ_u integrable nonnegative function $g - \min(f, g)$ takes the value 0, and (by the condition H), $\frac{1}{u}\sigma[g - \min(f, g)] \to 0$. Hence, the nontangential limit $(v - h)/u = 0$, μ_u a.e. on E_2. In particular, we conclude that at all those $\underline{t}$, the nontangential limit of h/u equals f, μ_u a.e. Since $h = \sigma(\min(f, g))$ and is u-bounded, we deduce that $\min(f, g) = f$, μ_u a.e., i.e., $\sigma(g) \geq \sigma(f)$.

This is true for all $w = \sigma(g)$ in the defining class of $S_1(f)$, and we conclude that $S_1(f) \geq \sigma(f) = S_0(f)$. The required result is therefore proved for $f \geq 0$ bounded and Borel on $\mathbb{T}^n$. In general, for any $f \geq 0$ Borel and μ_u-integrable, let $f_n = \min(f, n)$, for all n. Then,

$$S_1(f) \geq S_1(f_n) = S_0(f_n) \nearrow S_0(f) \geq S_1(f).$$

The proof is complete.

We include the following result for the sake of completeness.

Theorem 9. *Let $f : \mathbb{D}^n \to \mathbb{R}$ be a continuous function. Let $\tilde{f}(\underline{t}) = $ nontangential $\lim \sup f(\underline{z})$ as $\underline{z} \to \underline{t}$, for all $\underline{t} \in \mathbb{T}^n$. Then, $\tilde{f}$ is a Borel function.*

Proof. Let θ be a fixed number $0 < \theta < \pi/2$. Let $R_{\theta,\delta}(\underline{t})$ be the nontangential domain in $\mathbb{D}^n$ with vertex $\underline{t}$, which is the product of the Stolz domains $S_j(\theta, \delta)$ with vertex t_j, of semivertical angle θ and cut off by a circle of radius δ centered at t_j. Let m be a sufficiently large number so that the closed polydisc $\mathbb{D}^n_m$ of radius $1 - \frac{1}{m}$ has nonvoid intersection with $R_{\theta,\delta}$. From the fact that f is uniformly continuous on $\mathbb{D}^n_m$ we deduce easily that g_m defined by $g_m(\underline{t}) = \sup f(\underline{z})$ for $z \in R_{\theta,\delta}(\underline{t}) \cap \mathbb{D}^n_m$ is (uniformly) continuous on T^n. It is clear that if $\ell_\theta(\underline{t}) = \lim_{\delta \to 0}[\lim_{m \to \infty} g_m(\underline{t})]$ then $\ell_\theta(\underline{t}) = \lim \sup f(\underline{z})$ as $\underline{z} \to \underline{t}$ and $\underline{z}$ staying in the product of Stolz domain of angle θ. Hence, ℓ_θ is a Borel function; in fact, with very minimal restrictions the limit of a decreasing sequence of lower semicontinuous functions on $\mathbb{T}^n$. As we have remarked earlier the nontangential $\lim \sup$ of f is the increasing limit of the functions ℓ_θ as $\theta \uparrow \pi/2$. The theorem is proved.

Theorem 10. *Let $u > 0$ be an n-harmonic function on $\mathbb{D}^n$ that verifies the condition (H). Let w be the integral of the product of Poisson kernels relative to the (signed) measure $f \, d\mu_u$ where f is a μ_u-integrable Borel function on $\mathbb{T}^n$. Then the nontangential limit of w/u exists μ_u a.e. and equals f.*

Proof. Clearly, it is enough to prove the result for $f \geq 0$. Let $\tilde{f}$ be the nontangential $\lim \sup$ function associated to $\sigma(f)/u$. By the last theorem, $\tilde{f}$ is a Borel function and we know that $\tilde{f} \geq n - t \lim \inf \sigma(f)/u \geq f$, μ_u a.e. However, by Theorems 8 and 7

$$\sigma(f) \geq S_1(\tilde{f}) = S_0(\tilde{f}) = \sigma(\tilde{f}) \geq \sigma(f).$$

This clearly implies that $f = \tilde{f}$, μ_u a.e. The proof is complete.

Theorem 11. *Let u_j, $j = 1, 2$ be a positive n-harmonic function on $\mathbb{D}^n$ with the canonical representing Borel measure μ_j on $\mathbb{T}^n$ such that the condition (H) is verified by u_j. Then the following are results concerning the corresponding properties of $(u_1 + u_2)$.*

(a) *Every $(u_1 + u_2)$-bounded n-harmonic function $\mathbb{D}^n$ has a nontangential limit for $(\mu_1 + \mu_2)$ almost every element of $\mathbb{T}^n$ iff the nontangential limit of u_j/u_k exists as a real number for μ_k almost every element of $\mathbb{T}^n$, for $j, k = 1, 2$.*

(b) *Let ν_j be the singular part of the measure μ_j relative to μ_k. Then $(u_1 + u_2)$ satisfies the property (H) iff for all positive $(\mu_1 + \mu_2)$-integrable Borel functions f, $(\int f \mathbb{P} \, d\nu_j)/u_k$ has nontangential limit 0 for μ_k almost every element of $\mathbb{T}^n$ for $j = 1, 2$ and $k \neq j$.*

Proof. (a) Let g_j be the Radon-Nikodym derivative of the absolutely continuous part of μ_j relative to μ_k. Then, $d\mu_j = g_j d\mu_k + d\nu_j$. Assume that u_j/u_k has the required property. Let E be any Borel set. Then

$$\frac{(u_1 + u_2)_E}{u_1 + u_2} = \frac{1}{u_k}\left[(u_k)_E + \int_E g_j \mathbb{P}d\mu_k + \int \mathbb{P}d\nu_j\right] \Big/ \left[1 + \frac{u_j}{u_k}\right],$$

where $\mathbb{P}$ is the product of the Poisson kernels. For $k \neq j$, $0 \leq \int_E \mathbb{P}d\nu_j/u_k \leq \int \mathbb{P}d\nu_j/u_k$ tends to zero μ_k-a.e. and the function g_j being clearly Borel and μ_k-integrable we conclude that the above quotient has a nontangential limit equal to zero μ_k almost everywhere in the complement of E. This being true for $k = 1, 2$ we conclude that the condition of Theorem 3 is fulfilled for $(u_1 + u_2)$.

Conversely, the assumption on $(u_1 + u_2)$ clearly implies that $u_j/(u_j + u_k)$ has nontangential limit $(\mu_1 + \mu_2)$ almost everywhere for $j = 1, 2$. This establishes in particular that u_k/u_j has nontangential limit (including possibly $+\infty$) $(\mu_1 + \mu_2)$ almost everywhere. Using the fact that this limit is indeed the iterated nontangential limit we conclude by the Lemma 1 that the nontangential limit (function) of u_k/u_j is the equal to g_k (μ_j almost everywhere).

Let us now prove (b). Let g_j, ν_j be as above. Suppose $(u_1 + u_2)$ satisfies (H) and $f \geq 0$ and belongs to $L^1(\mu_1 + \mu_2)$. Let F_j be a Borel set carrying the measure ν_j. We have,

$$n - t.\lim \frac{\int f\mathbb{P}d\nu_j}{u_k} = n - t.\lim \left(\int_{F_j} f\mathbb{P}(d\mu_1 + d\mu_2)\right)\frac{1}{u_k}, \qquad \mu_k \text{ a.e.},$$

since $\mu_k(F_j) = 0$ and $\mu_j/F_j = \nu_j$. Clearly, by dividing the numerator and denominator of the above quotient by $(u_1 + u_2)$ and taking the limit, we get that this nontangential limit function equals $(f\chi_{F_j})(1 + g_j)$. Hence it is zero μ_k almost everywhere. This is true for $k = 1, 2$.

Conversely, suppose u_1 and u_2 verify all the listed conditions and f be in $L^1(\mu_1 + \mu_2)$ and $f \geq 0$. We have

$$\tau(f) = \int f\mathbb{P}(d\mu_1 + d\mu_2) = \int f\mathbb{P}d\mu_k + \int f\mathbb{P}g_j d\mu_k + \int f\mathbb{P}d\nu_j.$$

We can express $\tau(f)/(u_1 + u_2)$ as $(\tau(f)/u_k)(u_k/u_1 + u_2)$ and consider the nontangential limits of $\tau(f)/u_k$. It is trivial to see that nontangential limit of u_j/u_k exists μ_k almost everywhere for $j, k = 1, 2$. Clearly, the first two terms in the sum tend respectively to f and fg_j, μ_k almost everywhere. The last term tends to zero μ_k almost everywhere by our hypothesis. In particular, $\tau(f)/(u_1 + u_2)$ tends to zero at all points where $f = 0$, μ_k almost everywhere. This is true for $k = 1, 2$. Thus we have shown that $(u_1 + u_2)$ satisfies the property (H). The proof is complete.

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DEPARTMENT OF MATHEMATICS & STATISTICS, McGILL UNIVERSITY, 805 SHERBROOKE W, MONTREAL H3A 2K6, CANADA