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ILLUMINATION FOR UNIONS OF BOXES IN R^d

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ABSTRACT. Let S be a finite union of boxes (polytopes whose edges are parallel to the coordinate axes) in R^d. If every two vertices of S are clearly illumined by some common translate of the box T, then there is a translate of T that clearly illumines every point of S. A similar result holds when appropriate boundary points of S are illumined (rather than clearly illumined) by translates of box T.

1. INTRODUCTION

Let S be a set in R^d. For points x, y in S, we say x *sees* y via S (x is *visible* from y via S) if and only if the corresponding segment $[x, y]$ lies in S. Similarly, x is *clearly visible* from y via S if and only if there is some neighborhood N of x such that y sees via S each point of $N \cap S$. When $T \subseteq R^d$ and $S' \subseteq S$, T is said to *illumine* S' via S (T is an *illuminator* for S' via S) if and only if each point of S' sees via S some point of T. Likewise, set T *clearly illumines* S' via S if and only if each point of S' is clearly visible via S from some point of T. Finally, set S is *starshaped* if and only if there is some point p in S such that p sees via S each point of S.

A theorem by Krasnosel'skii [7] states that if S is a nonempty compact set in R^d, S is starshaped if and only if every $d+1$ points of S are visible via S from a common point. The proof uses the familiar Helly theorem: For $\mathscr{F}$ a family of compact convex sets in R^d, $\bigcap\{F : F \text{ in} \mathscr{F}\} \neq \varnothing$ if and only if every $d+1$ members of $\mathscr{F}$ have a nonempty intersection. There is an analogue of Helly's theorem by Klee [6] that holds when every $d+1$ members of F contain a common translate of some compact set T. It is reasonable, therefore, to look for analogues of Krasnosel'skii's theorem whenever every $d+1$ points of set S are visible from a common translate of a suitable set T. In such analogues, instead of showing that S is starshaped, the idea is to show that S has a convex illuminator (which is again a translate of T). Some recent results in this area appear in a paper by Bezdek, Bezdek, and Bisztriczky [1] and also in [2].

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In this paper, the illumination problem is studied with the additional condition that set S be a finite union of boxes in R^d; that is, S will be a finite union of polytopes whose edges are parallel to the coordinate axes. Toussaint and El-Gindy [11] have proved that when S is a finite union of boxes in R^2, S is starshaped if and only if every two points of S are visible via S from a common point. The following illumination analogue will be established in R^d: For S a finite union of boxes in R^d, every two vertices of S are clearly illumined by some translate of box T if and only if there is a translate of T that clearly illumines every point of S. A similar result is obtained with "clearly illumined" replaced by the weaker term "illumined."

The following familiar notation and terminology will be used: $\operatorname{aff} S$, $\operatorname{cl} S$, $\operatorname{int} S$, $\operatorname{rel\,int} S$, and $\operatorname{bdry} S$ will denote respectively the affine hull, closure, interior, relative interior, and boundary, for set S. When S is convex, $\dim S$ will be the dimension of S. The distance from point x to point y will be denoted $\operatorname{dist}(x, y)$. Hyperplane H is said to *separate* sets S and T in R^d if and only if S and T lie in opposite closed halfspaces determined by H. Point z is said to be *beyond* hyperplane H from set T if and only if H separates $\{z\}$ and T and $z \notin H$. The reader is referred to Valentine [12] and Lay [8] for a discussion of these concepts, to Danzer, Grünbaum, and Klee [4] for a survey of Helly-type results, and to O'Rourke [10] and Lenhart et al. [9] for applications of visibility concepts to problems in computational geometry.

2. The results

Let $\mathscr{P}$ be a finite collection of boxes (polytopes whose edges are parallel to the coordinate axes) in R^d, and let $S = \bigcup\{P : P \text{ in } \mathscr{P}\}$. We define a point v in $\operatorname{bdry} S$ to be a *vertex* of S if and only if $\{v\}$ is a zero-dimensional intersection of faces of polytopes in $\mathscr{P}$; that is, a vertex of S is a set $\bigcap\{F_i : 1 \le i \le j\} \subseteq \operatorname{bdry} S$, where each F_i is a face of some polytope P_i in $\mathscr{P}$ and where $\dim(\bigcap\{F_i : 1 \le i \le j\}) = 0$. We call set F a $(k-1)$-*facet* (a facet) of S if and only if for some k-dimensional box P in $\mathscr{P}$ and some $(k-1)$-face G of P, F is the closure of a component of $(\operatorname{rel\,int} G) \sim (\operatorname{int} S)$.

The first theorem concerns the case in which members of $\mathscr{P}$ are fully d-dimensional.

Theorem 1. *Let $\mathscr{P}$ be a finite collection of d-dimensional boxes in R^d, and let $S = \bigcup\{P : P \text{ in } \mathscr{P}\}$. If every two vertices of S are clearly illumined by some common translate of the box T, then there is a translate of T that clearly illumines every point of S.*

Proof. The argument is very easy if $d \le 1$, so assume $d \ge 2$. For each vertex x of S, define $\mathscr{F}_x$ to be the collection of facets F of boxes in $\mathscr{P}$ satisfying $\dim(N \cap F \cap \operatorname{bdry} S) = d-1$ for every neighborhood N of x. For F in $\mathscr{F}_x$, let P_F be the corresponding member of $\mathscr{P}$. Let $H_F = \operatorname{aff} F$, and label the associated closed halfspaces so that $P_F \subseteq \operatorname{cl}(H_F)_1$. Finally, define $C_x = \bigcap\{\operatorname{cl}(H_F)_1 : F \text{ in } \mathscr{F}_x\} \cap M$, where M is a minimal box containing S. Certainly C_x is a box, and if point s clearly illumines x (via S), then $s \in C_x$.

Now for each x, define $D_x = \{y : y + T \text{ meets } C_x\}$. Standard arguments [8, 12] show that D_x is convex and compact. We assert that D_x is a box as well. It suffices to show that for points $u = (u_1, \dots, u_d)$ and $v = (v_1, \dots, v_d)$ in D_x, the minimal box with vertices u and v is in D_x. In fact, since D_x is convex, it suffices to show that the vertices of this box are in D_x. Let $w = (w_1, \dots, w_d)$ be one of these vertices. Then $w_i \in \{u_i, v_i\}$, $1 \le i \le d$. For convenience, assume $w_i = u_i$ for $1 \le i \le j$ and $w_i = v_i$ for $j+1 \le i \le d$.

Since u is in D_x, $u+T$ meets C_x; that is, for some point $t = (t_1, \dots, t_d)$ in T, $u + t = (u_1 + t_1, \dots, u_d + t_d)$ is in C_x. Likewise, for some $t' = (t'_1, \dots, t'_d)$ in T, $v + t' = (v_1 + t'_1, \dots, v_d + t'_d) \in C_x$. Since C_x is a box, $b = (u_1 + t_1, \dots, u_j + t_j, v_{j+1} + t'_{j+1}, \dots, v_d + t'_d)$ is in C_x. Similarly, since T is a box, $(t_1, \dots, t_j, t'_{j+1}, \dots, t'_d)$ is in T, and $b = (w_1, \dots, w_d) + (t_1, \dots, t_j, t'_{j+1}, \dots, t'_d)$ is in $w + T$. Hence, $b \in C_x \cap (w + T) \neq \varnothing$, and $w \in D_x$, finishing the argument. The assertion is established.

Consider the finite collection of boxes $\{D_x : x \text{ a vertex of } S\}$. By hypothesis, every two vertices of S are clearly illumined by a common translate of T. Hence, every two of the D_x sets have a nonempty intersection. By a theorem of Danzer and Grünbaum [3], $\bigcap\{D_x : x \text{ a vertex of } S\} \neq \varnothing$. Select point y_0 in this intersection.

We shall see that each point s in S is clearly illumined by a point of $y_0 + T$. Assume that $s \notin y_0 + T$, for otherwise, the result is trivial. Let $a = (a_1, \dots, a_d)$ denote the point of $y_0 + T$ nearest s. Without loss of generality, assume that s is the origin Θ and that $a_i \ge 0$, $1 \le i \le d$. Let N be a box neighborhood of s disjoint from $y_0 + T$ such that N meets only those boxes in $\mathscr{P}$ that contain Θ. Let C' be such a box, and let $C = C' \cap N$. Finally, let A be a minimal box containing $C \cup \{a\}$. We will show that $A \subseteq S$ and hence a illumines each point of C via S.

If $a \in C$, there is nothing to show, so assume that $a \notin C$. Let B be any box containing C and contained in $A \cap S$ for which $\operatorname{dist}(B, a)$ is minimal, and let $(b_1, \dots, b_d)$ denote the point of B for which each component b_i is as large as possible. (See Figure 1.)

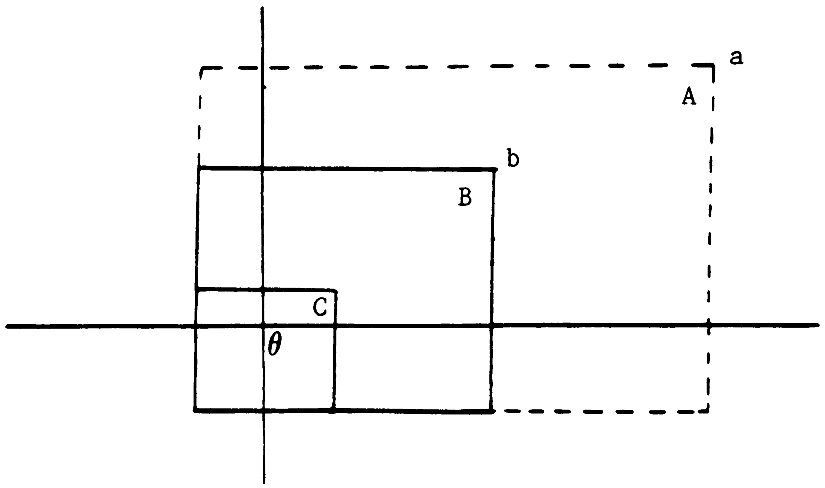

FIGURE 1

We assert that $a = b$. Suppose on the contrary that for some i, say for $i = 1$, $a_1 \neq b_1$. Then $b_1 < a_1$. By our choice of b, there is a facet F of S supported by hyperplane $x_1 = b_1$ such that no point of rel int F is illumined by any point in the halfspace $\{(x_1, \dots, x_d) : x_1 > b_1\}$. Moreover, the set cl(rel int F) $= F$ necessarily contains a vertex of S: In case F is also a facet of some box P_F in $\mathscr{P}$, obviously each vertex of P_F in F will be a vertex of S. Otherwise, for a facet G of some box in $\mathscr{P}$, F is the closure of a component of (rel int G) $\sim$ (int S). The set $J = \mathrm{cl}[G \cap (\mathrm{int}\, S)]$ is a finite union of $(d-1)$-dimensional boxes lying in G, and $F \cap J \neq \varnothing$ is a finite union of $(d-2)$-dimensional boxes. Any vertex of one of these $(d-2)$-dimensional boxes is necessarily a vertex of S in F. For v a vertex of S in $F = \mathrm{cl}(\mathrm{rel\ int}\, F)$, each neighborhood of v meets rel int F, and hence v cannot be clearly visible from any point in the halfspace $\{(x_1, \dots, x_d) : x_1 > b_1\}$. Since $y_0 + T \subseteq \{(x_1, \dots, x_d) : x_1 > b_1\}$, this implies that $(y_0 + T) \cap C_v = \varnothing$ and $y_0 \notin D_v$, contradicting our choice of y_0. Our supposition is false, $a = b$, $A \subseteq S$, and a illumines each point of C via S. Since each point of $N \cap S$ belongs to some box C, this implies that a illumines each point of $N \cap S$ and a clearly illumines Θ. Hence $s = \Theta$ is clearly illumined via S from a point of $y_0 + T$. The theorem is established.

Corollary 1 extends Theorem 1 to boxes having arbitrary dimensions.

Corollary 1. *Let $\mathscr{P}$ be a finite collection of boxes in R^d, and let $S = \bigcup\{P : P \text{ in } \mathscr{P}\}$. If every two vertices of S are clearly illumined by some common translate of the box T, then there is a translate of T that clearly illumines every point of S.*

Proof. If every box is fully d-dimensional, then the proof follows by Theorem 1. Otherwise, for each n, let B_n be the d-cube having edge length $\frac{1}{n}$ and centered at the origin. Let $S_n = S + B_n$, $T_n = T + B_n$. It is easy to show that sets S_n and T_n satisfy the hypothesis of Theorem 1. Hence there is a translate $y_n + T_n$ of T_n that clearly illumines each point of S_n, $n \geq 1$. Since $\{y_n\}$ is bounded, it has a convergent subsequence. Without loss of generality, say $\{y_n\}$ converges to y_0.

We assert that $y_0 + T$ satisfies the corollary. Let $s \in S$. (Then $s \in S_n$ for every $n \geq 1$.) As in the proof of Theorem 1, assume that $s \notin y_0 + T$. For $n \geq 1$, let a_n be the point of $y_n + T_n$ nearest s, and again without loss of generality assume $\{a_n\}$ converges to a_0. Certainly $\{y_n + T_n\}$ converges to $y_0 + T$ relative to the Hausdorff metric, so $a_0 \in y_0 + T$.

We will show that a_0 clearly illumines s via S. Let N be a box neighborhood of s disjoint from $y_0 + T$ and meeting only those boxes in $\mathscr{P}$ that contain s. Let C' be such a box, $C = C' \cap N$, $C'_n = C' + B_n$, and let $C_n = C'_n \cap N$. By the argument in Theorem 1, a_n illumines via S_n each point of C_n, and hence a_n illumines via S_n each point of C. Then certainly a_0 illumines via S each point of C. Since each point of $N \cap S$ is in some box C, a_0 illumines via S each point of $N \cap S$, and a_0 clearly illumines s via S. This finishes the proof of the corollary.

The following example shows that the notion of "clearly illumined" cannot be replaced by "illumined" in Theorem 1.

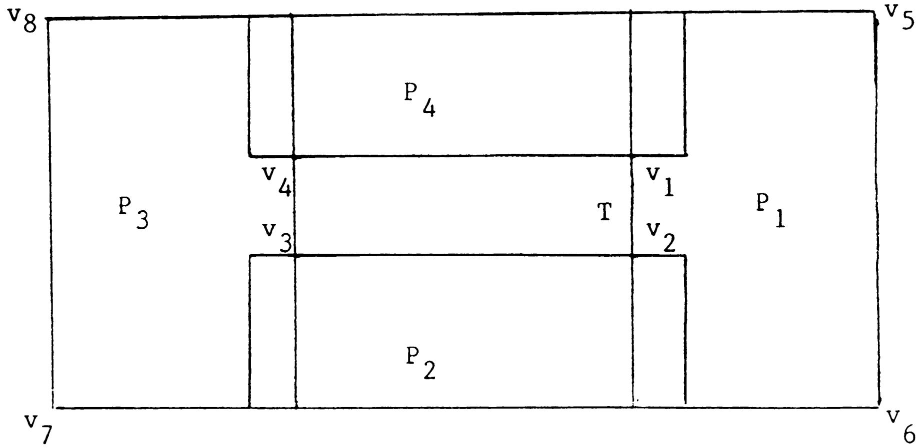

FIGURE 2

Example. Let $\mathscr{P} = \{P_i : 1 \leq i \leq 4\}$ be the collection of rectangles in Figure 2, and let $S = \bigcup\{P_i : 1 \leq i \leq 4\}$. Also let T be the vertical segment $[v_1, v_2]$. Every vertex v_i of S, $1 \leq i \leq 8$, is illumined by some point of T. Yet there is no translate of T that illumines every point of S.

However, if we replace "clearly illumined" by "illumined" and "vertices of S" by an appropriate set of boundary points of S, we obtain the following analogue of Theorem 1.

Theorem 2. *Let S be a finite union of boxes in R^d. For each facet F of S, choose some $x \in \operatorname{rel\,int} F$, and let B denote the collection of points x selected. If every two members of B are illumined by some common translate of box T, then there is a translate of T that illumines every point of S.*

Proof. For the moment, assume that each box is fully d-dimensional. For each x in B, let F_x be the corresponding facet with $H = \operatorname{aff} F_x$. Label the halfspaces determined by H so that the points that clearly illumine x lie in $\operatorname{cl}(H_F)_1$, and define $C_x = \operatorname{cl}(H_F)_1 \cap M$, where M is a minimal box containing S. As in Theorem 1, C_x is a box, and if $D_x = \{y : y + T \text{ meets } C_x\}$, then D_x is a box as well. Moreover, a simplified version of the previous argument shows that for $y_0 \in \bigcap\{D_x : x \text{ in } B\} \neq \varnothing$, each point of S is illumined via S by some point of $y_0 + T$.

In case some box is not d-dimensional, then an argument like the one in Corollary 1 finishes the proof.

Comment. Of course the converse is immediate in each of Theorem 1, Corollary 1, and Theorem 2.

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