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GLOBAL INVERTIBILITY OF EXPANDING MAPS

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ABSTRACT. We prove a global inversion theorem in reflexive Banach spaces utilizing a recent generalization of the interior mapping theorem. As a corollary, we provide, under a mild approximation property, a positive answer to an open problem that was stated by Nirenberg. We also establish global invertibility of an α-expanding Fréchet differentiable map in Banach space under the assumption that the logarithmic norm of the derivative is negative.

The study of nonlinear operator equations of the form $F(x) = y$, where F is a map from a Banach space X into a Banach space Y, is a central topic in nonlinear functional analysis and its applications. Typically, it is important to find conditions on F that guarantee the existence of a solution for each $y \in Y$. It is also important, particularly when approximate and iterative methods are involved, to determine conditions that ensure uniqueness and continuous dependence of solutions. In other words, a central problem is to find tractable conditions for the map F to be a *global homeomorphism.*

Problems of global inversions have been studied by several authors, e.g., Browder [1], Cristea [4], Nirenberg [12], Plastock [13], and Rădulescu [14, 15]; see references cited therein for earlier contributions. See also the recent books by Deimling [5] and Zeidler [16]. Along this line, Nirenberg [12, p. 175] posed the following interesting problem:

Let H be a Hilbert space and $T: H \to H$ be a continuous expanding map (i.e., $\|Tx - Ty\| \geq \|x - y\|$). Let $T(0) = 0$ and suppose that T maps a neighborhood of the origin onto a neighborhood of the origin. Does T map H onto H?

A partial answer to this problem was given by Chang and Shujie [3]. They prove the surjectivity of $T: X \to Y$ in the case when Y is reflexive, under the additional assumptions that T is Fréchet differentiable and

$$\limsup_{x \to x_0} \|T'(x) - T'(x_0)\| < 1 \quad \text{for all } x_0 \in X .$$

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Morel and Steinlein [10] gave an example of a map T in a nonreflexive Banach space with properties required in Nirenberg's problem, which is not onto. Thus Nirenberg's problem has no positive answer in the general framework of Banach spaces.

The aim of the present paper is to prove global inversion theorems for expanding maps in Banach spaces. As a consequence we provide a positive answer to the problem stated by Nirenberg in reflexive Banach spaces under a mild approximation property, but without requiring differentiability.

For a fixed element $y \in Y$, $[0, y]$ will denote the ray extending from 0 to y, i.e., $[0, y] := \{ty : 0 \leq t \leq 1\}$. As usual, $L(X, Y)$ denotes the space of all bounded linear operators on X into Y and $B(x_0, r) := \{x \in X : \|x - x_0\| < r\}$.

We now state the main theorem of this paper.

Theorem 1. *Let X and Y be reflexive Banach spaces and $F: X \to Y$ be a continuous map with the following properties*:

(i) $F(0) = 0$.

(ii) *Every point of X has a neighborhood that is mapped one to one by F.*

(iii) *For each $y \in Y$, there exists a constant $\alpha(y) > 0$ such that*

$$\alpha(y)\|x_1 - x_2\| \leq \|F(x_1) - F(x_2)\|$$

for all $x_1, x_2 \in F^{-1}([0, y])$.

(iv) *For each $x_0 \in X$ there exist constants $r > 0$, $\rho > 0$, $\beta \geq 0$ (all depending on x_0) and a convex bounded subset $\mathscr{M}$ of $L(X, Y)$ (also depending on x_0), such that whenever $x \in B(x_0, r)$ and $h \in X$, there are $\varepsilon \in (0, 1]$ and $L \in \mathscr{M}$ fulfilling*

$$\|F(x - \varepsilon h) - F(x) + \varepsilon Lh\| \leq \varepsilon\beta\|h\|.$$

Moreover, for each $L \in \mathscr{M}$ and each $y \in Y$, there exists $x \in X$ such that

$$Lx = y \quad \text{and} \quad \|y\| \geq (\beta + \rho)\|x\|.$$

Then F is a global homeomorphism from X onto Y.

One of the principal approaches to establishing *global* inverse mapping theorems is to use an appropriate *local* inverse mapping theorem and to seek, if possible, a construction yielding a global extension. The novelty of the method used in this paper resides in carrying this extension along rays. For the *local invertibility* we shall invoke the following recent result of Fabian and Preiss [6], which is a generalization of the interior mapping theorem of Clarke and Pourciau (see [6] for details).

Lemma 2. *Let X and Y be reflexive Banach spaces; let $r > 0$, $\rho > 0$, and $\beta \geq 0$; let $F: X \to Y$ be a continuous mapping, and let $x_0 \in X$. Moreover, let there exist a convex bounded subset $\mathscr{M}$ of $L(X, Y)$ such that whenever $x \in B(x_0, r)$ and $h \in X$, there are $\varepsilon \in (0, 1]$ and $L \in \mathscr{M}$ fulfilling*

$$\|F(x - \varepsilon h) - F(x) + \varepsilon Lh\| \leq \varepsilon\beta\|h\|.$$

Finally, let us assume that the mappings from $\mathscr{M}$ are uniformly open in the sense that, for each $L \in \mathscr{M}$ and each $y \in Y$, there exists $x \in X$ such that

$$Lx = y \quad \text{and} \quad \|y\| \geq (\beta + \rho)\|x\|.$$

Then the open ball $B(F(x_0), \rho r)$ of center $F(x_0)$ and radius ρr is contained in $F(B(x_0, r))$.

Proof of Theorem 1. First we prove that F is an open map. Let U be an open subset of X, and choose any $y = F(x)$ with $x \in U$. Then there exists $r_1 > 0$ such that $B(x, r_1) \subset U$. Take $r = \min\{r_1, r_x\}$, where r_x is the number given by (iv). Then, by Lemma 2, we have that

$$B(F(x), \rho_x r) \subset F(B(x, r)) \subset F(U),$$

so $F(U)$ is an open subset of Y. Using this result and hypothesis (ii), we conclude that F is a local homeomorphism. Therefore, there exists a ball B about $F(0) = 0$ in Y and a continuous local inverse $g: B \to X$ of F with $g(0) = 0$.

Now, we continue the local inverse g along each ray from 0 as far out as possible. More precisely, for any element y of Y let $[0, y]$ denote the ray $\{ty : 0 \leq t \leq 1\}$, and let D be the set of all points $y \in Y$ such that there exists a continuous inverse g of F defined on the ray $[0, y]$ and satisfying $g(0) = 0$. It is known that the value of $g(y)$ depends uniquely on F and y, and that D is an open subset of Y. It is also known that the map F^{-1} defined on D by $F^{-1}(y) = g(y)$ is an inverse of F on D and that F^{-1} is continuous on D. For details, see John [8].

The crux of the proof is then to show that $D = Y$. Suppose this is not the case. Then, by the construction of D and since it is a nonempty open set, there exists $y \in \overline{D}$ such that $y \notin D$ and $[0, y) \subset D$. Let $\{y_n\}$ be a sequence in $[0, y)$ such that $\lim_{n\to\infty} y_n = y$, and denote $x_n = F^{-1}(y_n)$. Hence, $x_n \in F^{-1}([0, y])$ for all $n \geq 1$, and consequently

$$\|x_n - x_m\| \leq \alpha(y)\|F(x_n) - F(x_m)\|$$

for all n and m. Therefore, $\{x_n\}$ is a Cauchy sequence in X, so there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Thus, $F(x) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} y_n = y$, i.e., $y \in F(x)$. Now, because of the local invertibility of F , there exist $r > 0$ and a continuous map $g: B(y, r) \to g(B(y, r))$ with the following properties:

(a) $g(B(y, r))$ is an open subset of X,
(b) g is a continuous inverse of F, and
(c) $g(y) = x$.

On the other hand, there exists a natural number $N \geq 1$ such that $x_n \in g(B(y, r))$ for all $n \geq N$. Then, by injectivity of the map F on $g(B(y, r))$, we have that

$$g(y_n) = x_n = F^{-1}(y_n)$$

for all $n \geq N$. Let $k > N$ and consider the element y_k. It is obvious that

$$y_k \in [0, y) \cap B(y, r) \subset D,$$

so there exists a continuous inverse G_1 of F defined on the segment $[0, y_k]$ such that

$$G_1(y_k) = F^{-1}(y_k) = g(y_k).$$

Define the map $H: [0, y] \to X$ by

$$H(z) = \begin{cases} G_1(z) & \text{if } z \in [0, y_k], \\ g(z) & \text{if } z \in [y_k, y]. \end{cases}$$

It is obvious that H is a continuous inverse of F defined on $[0, y]$, and moreover, $H(0) = 0$. Therefore, $y \in D$, which is a contradiction. Thus $D = Y$, and the proof is completed.

Definition. Let X and Y be Banach spaces. A map $F: X \to Y$ is called α-expanding if there exists $\alpha > 0$ such that

$$\|F(x) - F(y)\| \geq \alpha\|x - y\|$$

for all $x, y \in X$ (α is called an expanding constant).

The following corollary provides a positive answer to the problem of Nirenberg.

Corollary 3. *Let X and Y be reflexive Banach spaces and $F: X \to Y$ be a continuous α-expanding map with expanding constant α and $F(0) = 0$. Suppose that for each $x_0 \in X$ there exist real numbers $r > 0$, $\rho > 0$, $\beta \geq 0$, (all depending on x_0) and a convex bounded subset $\mathscr{M}$ of $L(X, Y)$ (also depending on x_0) such that whenever $x \in B(x_0, r)$ and $h \in X$, there are $\varepsilon \in (0, 1]$ and $L \in \mathscr{M}$ fulfilling*

$$\|F(x - \varepsilon h) - F(x) + \varepsilon L h\| \leq \varepsilon\beta\|h\|.$$

Finally, let us assume that for each $L \in \mathscr{M}$ and each $y \in Y$, there exists $x \in X$ such that

$$Lx = y \quad \textit{and} \quad \|y\| \geq (\beta + \rho)\|x\|.$$

Then F is global homeomorphism from X onto Y.

Proof. Since F is an α-expanding map, F is one to one, so conditions (ii) and (iii) of Theorem 1 are satisfied.

Remark. Note that in Corollary 3 we do not explicitly *assume* that F maps a neighborhood of zero onto a neighborhood of zero; this follows from the approximation property that is assumed in the corollary. Moreover, under this approximation property, Corollary 3 answers positively the problem of Nirenberg in reflexive Banach spaces, not just Hilbert spaces, for α-expanding maps. Finally note that in Theorem 1, the map F is only required to be α-expanding on the inverse image of rays.

A simple illustration of the last corollary is given in the following example. The idea of this example is to illustrate the type of argument that one should use in order to apply Theorem 1, even though the example can be directly solved by a direct method.

Example. Let $X = Y = \Re^2$ and $F: X \to Y$ be defined by

$$F(x, y) = (2x + |x|, y).$$

Here $\Re^2$ is endowed with the Euclidean norm. Obviously, F is a continuous function. Since for each $x_1, x_2 \in \Re$ the inequality

$$|x_1 - x_2| \leq |\, 2x_1 - 2x_2 + |x_1| - |x_2|\, |$$

holds, we have that

$$\begin{aligned}\|(x_1, y_1) - (x_2, y_2)\|^2 &\leq (2x_1 - 2x_2 + |x_1| - |x_2|)^2 + (y_1 - y_2)^2 \\ &= \|F(x_1, y_1) - F(x_2, y_2)\|^2,\end{aligned}$$

so F is an expanding map. First consider $z = (x_0, y_0)$ with $x_0 \neq 0$, and define the set $\mathscr{M}_z$ by $\mathscr{M}_z = \{L_z\}$, where $L_z\colon \Re^2 \to \Re^2$ is defined by

$$L_z(x, y) = \begin{pmatrix} 2 + \operatorname{sgn}(x_0) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

For $z = (0, y)$ let $\mathscr{M}_z$ be defined by $\mathscr{M}_z = \{ L_a : 1 \leq a \leq 3 \}$, where $L_a\colon \Re^2 \to \Re^2$ is defined by

$$L_a(x, y) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

It is not difficult to prove that $\mathscr{M}_z$ is a bounded convex subset of $\mathrm{L}(\Re^2, \Re^2)$ for each $z \in \Re^2$. Moreover, we have that for each $z \in \Re^2$ and $L \in \mathscr{M}_z$, $L^{-1} \in \mathrm{L}(\Re^2, \Re^2)$ and $\|L^{-1}\| \leq 1$. It is also easy to prove that for each $z \in \Re^2$ and each $\beta > 0$, there exists $r > 0$ such that for any $z_1, z_2 \in B(z, r)$, there exists $L \in \mathscr{M}_z$ with the property

$$\|F(z_1) - F(z_2) - L(z_1 - z_2)\| \leq \beta \|z_1 - z_2\|.$$

Finally, for each $z \in \Re^2$ take $0 < \beta_z < 1$, $\rho_z = r - \beta_z$, and r_z the number given by the above result. It is easy to check that with this choice of $\mathscr{M}_z$, r_z, ρ_z, and β_z, the hypotheses of Corollary 2 are satisfied. Hence, F is a global homeomorphism.

We next establish a theorem that gives a positive answer to the problem of Nirenberg in Banach spaces under the assumption that F is Fréchet differentiable. For properties of the Fréchet differentials and the logarithmic norm, see [7, 11].

Theorem 4. *Let X be a Banach space and $F\colon X \to X$ be an α-expanding map. Suppose that F is Fréchet differentiable in X and the logarithmic norm $\mu(F'(x))$ of $F'(x)$ is strictly negative for all $x \in X$, where*

$$\mu(F'(x)) := \lim_{t \to 0^+} \frac{\|I + tF'(x)\| - 1}{t}.$$

Then F is a global homeomorphism.

Proof. First of all, it is easy to prove that the image of any continuous α-expanding map is closed. Let $x \in X$ and $0 < \varepsilon < -\mu(F'(x))$. Then, by definition of $\mu(F'(x))$, there exists δ_ε such that $\delta_\varepsilon < -1/(\mu(F'(x)) + \varepsilon)$ and

$$\frac{\|I + tF'(x)\| - 1}{t} \leq \mu(F'(x)) + \varepsilon < 0, \quad \text{provided } t \in (0, \delta_\varepsilon),$$

so

$$\|I + tF'(x)\| \leq 1 + t[\mu(F'(x)) + \varepsilon] \quad \text{for all } t \in (0, \delta_\varepsilon).$$

Since

$$0 < t < \delta_\varepsilon < \frac{1}{-(\mu(F'(x)) + \varepsilon)},$$

we have $0 < -t(\mu(F'(x)) + \varepsilon) < 1$ and $0 < 1 + t(\mu(F'(x)) + \varepsilon) < 1$. Therefore, $\|I + tF'(x)\| < 1$ for all $t \in (0, \delta_\varepsilon)$. By the spectral properties of bounded linear operators [9, Theorem 7.31], $[I - (I + tF'(x))]^{-1} = [-tF'(x)]^{-1}$ exists as bounded linear operator on the whole space X, for all $t \in (0, \delta_\varepsilon)$. Consequently, $F'(x)$ is invertible for all $x \in X$. Thus, since $F(X)$ is closed, by

Theorem 2 of [2], we have that $F(X) = Y$. Since F is an α-expanding map, F is one to one. Hence, F is a bijection. Moreover, by the inequality

$$\|F^{-1}(y_1) - F^{-1}(y_2)\| \leq \alpha^{-1}\|y_1 - y_2\|, \qquad y_1, y_2 \in Y,$$

we conclude that F is a global homeomorphism.

Theorem 5. *Let H be a Hilbert space and $F\colon H \to H$ be an α-expanding map. Suppose that F is Fréchet differentiable in H and either*

$$\inf_{\|h\|=1} \operatorname{Re}\langle F'(x)h, h\rangle > 0 \quad \textit{for all } x \in H,$$

or

$$\sup_{\|h\|=1} \operatorname{Re}\langle F'(x)h, h\rangle < 0 \quad \textit{for all } \ x \in H.$$

Then F is a global homeomorphism.

Proof. For a fixed $x \in H$, consider the selfadjoint transformation

$$T(x) := \tfrac{1}{2}(F'(x) + [F'(x)]^*)$$

where $[F'(x)]^*$ is the adjoint of $F'(x)$. Denote by $M(x)$ the greatest element of the spectrum of $T(x)$. Then, according to [7, p. 112], $M(x) = \mu(F'(x))$. Therefore, since $M(x) = \sup_{\|h\|=1}\langle T(x)h, h\rangle$, we have

$$\mu(F'(x)) = \sup_{\|h\|=1} \tfrac{1}{2}\langle F'(x)h + [F'(x)]^*h, h\rangle = \sup_{\|h\|=1} \operatorname{Re}\langle F'(x)h, h\rangle.$$

Now, by assumption the function $\operatorname{Re}\langle F'(x)(\cdot), \cdot\rangle$ has a constant sign on H. Denote $\rho = \operatorname{sign}\operatorname{Re}\langle F'(x)h, h\rangle$ and define the function $g\colon H \to H$ by $g(x) = -\rho F(x)$. It is obvious that g is an α-expanding Fréchet differentiable map in H. Moreover, for each $x \in H$,

$$\sup_{\|h\|=1} \operatorname{Re}\langle g'(x)h, h\rangle = \sup_{\|h\|=1} \{-\rho \operatorname{Re}\langle F'(x)h, h\rangle\} < 0.$$

Hence, $\mu(g'(x)) < 0$ for all $x \in H$. Therefore, by Theorem 4, g is a global homeomorphism, so F is a global homeomorphism from H onto itself.

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