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FINITELY GENERATED RADICAL IDEALS IN H^∞

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ABSTRACT. It is shown that a radical ideal in H^∞ is finitely generated if and only if it is a principal ideal generated by a Blaschke product having simple zeros.

INTRODUCTION

Let H^∞ be the algebra of all bounded analytic functions in the open unit disk $\mathbb{D}$ and let $A(\mathbb{D})$ be the disk algebra. Forelli [1] noticed that the proofs in §2 of [6] show that a radical ideal in the disk algebra is finitely generated if and only if it is generated by a finite Blaschke product having simple zeros. It is, therefore, a natural question to ask for a characterization of the finitely generated radical ideals in H^∞. It is the aim of this note to give a complete solution to this problem. First we present some definitions.

Let R be a commutative algebra with identity element. An ideal I in R is called a radical ideal if whenever f^n belongs to I for some $n \in \mathbb{N}$ f is also an element of the ideal I. It is an easy exercise in algebra to show that the class of radical ideals coincides with those ideals that are intersections of prime ideals.

An ideal I is called finitely generated if there exist elements $f_1, \ldots, f_N \in R$ such that

$$I = (f_1, \ldots, f_N) = \left\{ \sum_{i=1}^{N} g_i f_i : g_i \in R \right\}.$$

If N can be chosen to be one, then I is a principal ideal.

In [3, 6] two of the authors showed that a prime ideal in H^∞ is finitely generated if and only if it is generated by a single function of the form $z - z_0$ for some $z_0 \in \mathbb{D}$. The proofs depend on Hoffman's theory of the maximal ideal space of H^∞ and on some of his deep factorization theorems [5; 2, §8]. These methods, however, do not seem to work for the present setting of radical ideals. Surprisingly, the techniques developed here are much simpler than our previous methods. They do not depend on Hoffman's theory but still yield the stronger result.

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Radical ideals in H^∞

We begin with some definitions.

A sequence (z_n) in $\mathbb{D}$ is called an interpolating sequence (for H^∞) if for every bounded sequence (w_n) of complex numbers there exists a function $f \in H^\infty$ such that $f(z_n) = w_n$ for all $n \in \mathbb{N}$. A Blaschke product

$$B(z) = z^m \prod_{n=1}^{\infty} \frac{\bar{a}_n}{|a_n|} \frac{a_n - z}{1 - \bar{a}_n z}, \qquad a_n \in \mathbb{D}\backslash\{0\},$$

$$\sum_{n=1}^{\infty} 1 - |a_n| < \infty, \qquad m \in \mathbb{N} \cup \{0\},$$

is called interpolating if the zero sequence (a_n) of B is an interpolating sequence. In particular, interpolating Blaschke products have only simple zeros. Finite Blaschke products with simple zeros will be regarded as interpolating Blaschke products.

Let $M(H^\infty)$ denote the maximal ideal space of H^∞, that is, the space of nonzero multiplicative linear functionals on H^∞ supplied with the Gelfand topology. By identifying the functional "evaluation at $z_0 \in \mathbb{D}$" with the point z_0 itself, we may regard $\mathbb{D}$ as a subset of $M(H^\infty)$. By Carleson's famous corona theorem, $\mathbb{D}$ is dense in $M(H^\infty)$ [2, §8].

A point $m \in M(H^\infty)$, which belong to $\mathbb{D}$ or lies in the closure of an interpolating sequence $\{z_n\}$ in $\mathbb{D}$, is called a nontrivial point (see [5]). In particular, the maximal ideal $\operatorname{Ker} m$ associated with m then contains the interpolating Blaschke product b with zero sequence $\{z_n\}$. For a function f in H^∞, let $\|f\| = \sup\{|f(z)| : z \in \mathbb{D}\}$ be its norm and let $Z(f) = \{z \in \mathbb{D} : f(z) = 0\}$ denote its zero set.

Finally, recall that an ideal I in H^∞ is called free if $\bigcap_{f \in I} Z(f) = \varnothing$.

Our methods rely heavily on the well-known Riesz factorization theorems for H^∞ (see [4, §5; 2, §2]). They tell us that every function f in H^∞, f not identically zero, can be written as $f = Bg$, where B is a Blaschke product and g is a zero free function in H^∞. Also, if $h \in H^\infty$ vanishes at every zero of the Blaschke product B with the same order, then $h = Bk$ for some $k \in H^\infty$.

Lemma 1.1. *Let $I = (f_1, \dots, f_N)$ be a finitely generated radical ideal in H^∞. Assume that the functions $f_1, \dots, f_N$ have no common zero in $\mathbb{D}$. Let $h = Bg$ be an element of I, where B is an interpolating Blaschke product and $g \in H^\infty$. Then $g \in I$.*

Proof. Let $\{z_n\}$ denote the zero sequence of B and let $w_n = \sqrt[4]{\sum_{j=1}^N |f_j(z_n)|^2}$. Note that we have $w_n \neq 0$ for any n. Because $\{z_n\}$ is an interpolating sequence, there exists a function $f \in H^\infty$ with $f(z_n) = w_n$ and functions $h_j \in H^\infty$ with $h_j(z_n) = \bar{f}_j(z_n)$ for all n. Therefore we have $f^4 - \sum_{j=1}^N h_j f_j = kB$ for some $k \in H^\infty$. Multiplication by g yields $gf^4 = kh + \sum_{j=1}^N (h_j g) f_j \in I$. Because I is a radical ideal, we have $gf \in I$. Thus there exist $k_1, \dots, k_N \in H^\infty$ such that $gf = \sum_{j=1}^N k_j f_j$. Hence

$$|(gf)(z_n)| \le C \sum_{j=1}^N |f_j(z_n)| \quad \text{where } C = \max_{1 \le j \le N} \|k_j\|.$$

Hence by Cauchy-Schwarz's inequality,

$$|(gf)(z_n)|^4 \leq C^4 N^2 \left(\sum_{j=1}^{N} |f_j(z_n)|^2 \right)^2 .$$

This and our choice of $w_n = f(z_n)$ imply that the sequence (v_n) defined by

$$v_n = (g(z_n))^4 / \sum_{j=1}^{N} |f_j(z_n)|^2$$

is bounded by $C^4 N^2$. Thus there exists $F \in H^\infty$ with $F(z_n) = v_n$ for all n. Hence

$$\left(F \left(\sum_{j=1}^{N} h_j f_j \right) - g^4 \right) (z_n) = F(z_n) \sum_{j=1}^{N} |f_j(z_n)|^2 - (g(z_n))^4 = 0,$$

from which we conclude that $F \sum_{j=1}^{N} h_j f_j - g^4 = lB$ for some $l \in H^\infty$.

Multiplication by g yields

$$g^5 = gF \sum_{j=1}^{N} h_j f_j - l(Bg) \in I.$$

Because I is radical, we obtain $g \in I$. □

Lemma 1.2. *Let $I = (f_1, \dots, f_N)$ be a finitely generated ideal in H^∞, and let m be a nontrivial point in $M(H^\infty)$ such that $I \subset \operatorname{Ker} m$. Then there exist $i_0 \in \{1, \dots, N\}$, functions $g_j \in H^\infty$ $(j = 1, \dots, N-1)$, and interpolating Blaschke products b_j with $b_j \in \operatorname{Ker} m$ $(j = 1, \dots, N-1)$ such that $I = (b_1 g_1, \dots, b_{N-1} g_{N-1}, f_{i_0})$.*

Proof. If $m \in \mathbb{D}$, then $m(f) = f(z_0)$ for some $z_0 \in \mathbb{D}$ and all $f \in H^\infty$. Hence $f_j(z) = ((z - z_0)/(1 - \bar{z}_0 z)) g_j(z)$ $(j = 1, \dots, N)$, and the conclusion follows easily. So let $m \in M(H^\infty) \backslash \mathbb{D}$ and let $\{z_n\}$ be an interpolating sequence in $\mathbb{D}$ capturing m in its closure. Let

$$A = \{z_n : |f_1(z_n)| \leq |f_2(z_n)| \text{ and } z_n \in \{z_j : j \in \mathbb{N}\}\},$$
$$B = \{z_n : |f_1(z_n)| > |f_2(z_n)| \text{ and } z_n \in \{z_j : j \in \mathbb{N}\}\}.$$

Since $m \in \overline{A \cup B}$ and $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$, we have $m \in \overline{A}$ or $m \in \overline{B}$. Without loss of generality we may assume $m \in \overline{A}$. Then if $\{z_n'\}$ denotes the sequence of points in A, we have

$$|f_1(z_n')/f_2(z_n')| \leq 1 \quad \text{for all } n \in \mathbb{N} \text{ such that } f_2(z_n') \neq 0.$$

Since $\{z_n'\}$ is an interpolating sequence, we can choose a function $h \in H^\infty$ such that

$$h(z_n') = \begin{cases} f_1(z_n')/f_2(z_n') & \text{if } f_2(z_n') \neq 0, \\ 1 & \text{if } f_2(z_n') = 0. \end{cases}$$

Note that $f_2(z_n') = 0$ implies $f_1(z_n') = 0$. Hence $f_2(z_n')h(z_n') - f_1(z_n') = 0$. Therefore $f_2 h - f_1 = b_1 g_1$ for some $g_1 \in H^\infty$ and some interpolating Blaschke product b_1 with $b_1 \in \operatorname{Ker} m$. Hence $I = (b_1 g_1, f_2, \dots, f_N)$.

Now repeat the above process with f_2 and f_3. One of them can be replaced by $b_2 g_2$, where $b_2 \in \operatorname{Ker} m$ is an interpolating Blaschke product and where $g_2 \in H^\infty$. Continuing in this way, we see that any time we choose two generators, one of them can be replaced by a function of the form $b_j g_j$, where b_j is an interpolating Blaschke product with $b_j \in \operatorname{Ker} m$ and where $g_j \in H^\infty$. □

It is known that there exist free prime ideals P in H^∞ with the property that $Z(f) \cap Z(g) \neq \varnothing$ for every pair f and g in P (see [7]). The following result now shows that this cannot happen for free ideals that are finitely generated.

Lemma 1.3. *Let $I = (f_1, \dots, f_{N+1})$ be a finitely generated ideal in H^∞ satisfying $\bigcap_{i=1}^{N+1} Z(f_i) = \varnothing$. If f_j is any of the generators, then there exists a function f in I such that $Z(f) \cap Z(f_j) = \varnothing$.*

Proof. Without loss of generality, let $j = N+1$. If $Z(f_{N+1}) = \varnothing$, then we are done. So let $\{z_n\}$ denote the zero set of f_{N+1}. We claim that there exists a point $(\alpha_1, \dots, \alpha_N) \in \mathbb{C}^N$ such that

$$\sum_{i=1}^{N} \alpha_i f_i(z_n) \neq 0 \quad \text{for every } n \in \mathbb{N}.$$

Let

$$S_n = \left\{ (\alpha_1^{(n)}, \dots, \alpha_N^{(n)}) \in \mathbb{C}^N : \sum_{i=1}^{N} \alpha_i^{(n)} f_i(z_n) = 0 \right\}.$$

Because $f_j(z_n) \neq 0$ for some $j = j(n) \in \{1, \dots, N\}$, S_n is an $(N-1)$-dimensional vector subspace of $\mathbb{C}^N$. Hence S_n has $2N$-dimensional Lebesgue measure zero. Therefore $\bigcup_{n=1}^\infty S_n$ has measure zero. Thus there exists $(\alpha_1, \dots, \alpha_N) \in \mathbb{C} \setminus \bigcup_{n=1}^\infty S_n$. This proves our claim.

Now let $f = \sum_{i=1}^N \alpha_i f_i$. Then, by construction, $Z(f) \cap Z(f_{N+1}) = \varnothing$. Obviously, $f \in I$. □

Finally, we need the following version of Nakayama's lemma:

Lemma 1.4. *Let $I = (f_1 g_1, \dots, f_n g_n, g_{n+1})$ be a finitely generated ideal in a commutative ring R with identity. Also assume that $g_j \in I$ $(j = 1, 2, \dots, n)$. Then there exists some $a \in (f_1, \dots, f_n)$ such that $(1+a)I \subseteq (g_{n+1})$.*

Proof. Because $g_j \in I$ $(j = 1, \dots, n)$, there exists $x_j, x_{jk} \in R$ $(j = 1, \dots, n$, $k = 1, \dots, n)$ such that

$$(*) \qquad g_j = \sum_{k=1}^{n} x_{jk} f_k g_k + x_j g_{n+1} \qquad (j = 1, \dots, n).$$

Let A be the matrix

$$A = \begin{pmatrix} 1 - x_{11} f_1 & -x_{12} f_2 & \cdots & -x_{1n} f_n \\ -x_{21} f_1 & 1 - x_{22} f_2 & & \vdots \\ \vdots & & \ddots & \\ -x_{n1} f_1 & \cdots & & 1 - x_{nn} f_n \end{pmatrix},$$

g the vector $g = (g_1, \dots, g_n)^\intercal$, and b the vector $b = (x_1 g_{n+1}, \dots, x_n g_{n+1})^\intercal$. Let a_j denote the j-column of A. Then $(*)$ is equivalent to the system $Ag = b$. By Cramer's rule we have

$$(\det A) g_j = \det(a_1, \dots, a_{j-1}, b, a_{j+1}, \dots, a_n) := r_j \qquad (j = 1, \dots, n).$$

But $\det A$ has the form $1+a$ for some $a \in (f_1, \dots, f_n)$. Moreover, $r_j \in (g_{n+1})$ $(j = 1, \dots, n)$.

Because $I = (g_1, \dots, g_n, g_{n+1})$, we finally obtain the assertion $(1+a)I \subseteq (g_{n+1})$. □

We are now in a position to prove our main theorem.

Theorem 1.5. *A radical ideal in H^∞ is finitely generated if and only if it is a principal ideal generated by a Blaschke product having simple zeros.*

Proof. Let B be a Blaschke product having simple zeros, and let $f^n = hB$ for some $n \in N$ and $h \in H^\infty$. Then $B(z_0) = 0$ implies that $f(z_0) = 0$ for each zero z_0 of B. Hence $f \in (B)$. Thus (B) is a radical ideal. To prove the converse, let $I = (h_1, \dots, h_N)$ be a finitely generated radical ideal in H^∞. Let B denote the Blaschke product formed with the common zeros in $\mathbb{D}$ of the functions h_j (including multiplicities). Then $h_j = Bf_j$ for some functions $f_j \in H^\infty$ $(j = 1, \dots, N)$.

Claim 1. B has only simple zeros.

Proof. Suppose that B has a zero of order $n > 1$ at some point $z_0 \in \mathbb{D}$. Then $h_j(z) = (z - z_0)^n k_j(z)$ for $z \in \mathbb{D}$ $(j = 1, \dots, N)$, and there exists j_0 such that $k_{j_0}(z_0) \neq 0$. Since $[(z - z_0)k_{j_0}(z)]^n = h_{j_0} k_{j_0}^{n-1}(z) \in I$ and I is radical, $(z - z_0)k_{j_0}(z) \in I$. Hence $(z - z_0)k_{j_0}(z) = (z - z_0)^n \sum_{j=1}^N k_j(z) l_j(z)$ for some functions $l_1, \dots, l_N \in H^\infty$. Dividing, we get $k_{j_0}(z_0) = 0$, a contradiction.

Claim 2. The ideal $J = (f_1, \dots, f_N)$ is a free radical ideal.

Proof. By construction, J is a free ideal. Now let $f^n \in J$. Then $Bf^n \in I$. Since I is radical, we have $Bf \in I$. Thus $f \in J$.

Claim 3. $J = H^\infty$.

Proof. If $J \neq H^\infty$, then there exists, by the corona theorem [2, §8], a sequence (z_n) in $\mathbb{D}$ such that $\sum_{j=1}^N |f_j(z_n)| \to 0$. Because J is free, $|z_n| \to 1$, and we may assume (by passing to a subsequence if necessary) that (z_n) is an interpolating sequence [4, p. 204]. Let m be any cluster point of $\{z_n\}$. Then $J \subseteq \operatorname{Ker} m := M$. By Lemma 1.2 there exist $g_j \in H^\infty$ and interpolating Blaschke products b_j $(j = 1, \dots, N-1)$ with $b_j \in M$ such that

$$J = (b_1 g_1, \dots, b_{N-1} g_{N-1}, f_{i_0}) \quad \text{for some } i_0 \in \{1, \dots, N\}.$$

Lemma 1.1 implies that $g_j \in J$. Lemma 1.4 yields the existence of a function $a \in M$ such that

$$(1 + a)J \subseteq (f_{i_0}). \tag{1}$$

By Lemma 1.3 we can choose a function $f \in J$ such that $Z(f) \cap Z(f_{i_0}) = \varnothing$. Let $f_{i_0} = bF$ be the Riesz factorization of f_{i_0}, where b is a Blaschke product and F is zerofree in $\mathbb{D}$. Since J is a radical ideal, we have $b\sqrt{F} \in J$. Hence

$(1+a)b\sqrt{F} = h(bF)$ for some $h \in H^\infty$. This implies that $1+a = h\sqrt{F}$. Since $a \in M$, we see that $F \notin M$. Because $f_{i_0} \in M$, we have $b \in M$. Now let $f = CG$ be the Riesz factorization of f. By (1) there exists $k \in H^\infty$ with $(1+a)CG = k(bF)$. Since $Z(b) \cap Z(C) = \varnothing$ and G has no zeros in $\mathbb{D}$, we see that $(1+a)(z_0) = 0$ whenever $b(z_0) = 0$ (including multiplicities). Therefore

$$1 + a = bh \tag{2}$$

for some $h \in H^\infty$. Since a and b belong to M, relation (2) implies that $1 \in M$, which is a contradiction. Therefore $J = H^\infty$.

So we can conclude that $I = BH^\infty$. □

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