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A CONVERSE OF LOTZ'S THEOREM ON UNIFORMLY CONTINUOUS SEMIGROUPS

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ABSTRACT. We prove the following partial converse to a theorem of Lotz: If every C_0-semigroup on a Banach lattice E with quasi-interior point is uniformly continuous, then E is isomorphic to a $C(K)$-space with the Grothendieck property.

Let us say that a Banach space X has the *Lotz property* if every C_0-semigroup on X is uniformly continuous. In [L] it was shown by Lotz that every Grothendieck space with the Dunford-Pettis property has the Lotz property, after some special cases were obtained by several authors, notably Coulhon [C], Kishimoto-Robinson [KR], and Lotz himself. Examples of Grothendieck Dunford-Pettis spaces are l^∞ and $L^\infty(\Omega, \mu)$ and, more generally, σ-Dedekind complete $C(K)$-spaces. We will give a partial converse of Lotz's result, which uses the following lemma [Ne]. A strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on a Banach lattice E is called a *multiplication semigroup* if each $T(t)$ is a band-preserving operator.

Lemma 1. *Let E be a Banach lattice with quasi-interior point $u > 0$. Let $v \leq 0$ be arbitrary. Then there exists a multiplication semigroup on E with generator A such that $u \in D(A)$ and $Au = v$.*

Theorem 2. *Let E be a Banach lattice with quasi-interior point. Then the following assertions are equivalent*

(1) *E has the Lotz property,*
(2) *E has the Grothendieck property and the Dunford-Pettis property,*
(3) *E is isomorphic to a $C(K)$-space with the Grothendieck property.*

Proof. $(1) \Rightarrow (3)$. Let $u > 0$ be a quasi-interior point of E. Let $v \leq 0$ be arbitrary, and let A be the generator of a multiplication semigroup on E with $Au = v$. Such a semigroup exists by the lemma. This semigroup is uniformly continuous by assumption, whence A is a bounded band-preserving operator. By [W], such operators preserve ideals. Hence $v = Au \in E_u$, the ideal generated by u. If $v \in E$ is arbitrary, $v \in E_u$ follows from $v = v^+ - v^-$. We have

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shown that $E = E_u$, in other words, u is actually a strong order unit for E. By the Kakutani representation theorem, E is Banach lattice isomorphic to a $C(K)$ space. We still have to prove that E is Grothendieck. By [N] it suffices to show that E does not contain a complemented subspace isomorphic to c_0. If such a subspace exists, say $E = c_0 \oplus F$, however, then $T(t) \oplus \mathrm{id}_F$, where $(T(t)x)_n := e^{-nt}x_n$, defines a strongly continuous semigroup on E that is not uniformly continuous.

(3) $\Rightarrow$ (2). Every $C(K)$-space has the Dunford-Pettis property.

(2) $\Rightarrow$ (1). This follows from Lotz's theorem.

Remark. It follows from the theorem that every Grothendieck Dunford-Pettis lattice with a quasi-interior point is a $C(K)$-space.

Remark. In [Le] the so-called *surjective Dunford-Pettis property* is introduced. It is shown there that every Grothendiek space with the surjective Dunford-Pettis property has the Lotz property. Moreover, an example is constructed of a Grothendieck lattice with a weak order unit having the surjective Dunford-Pettis property but not the Dunford-Pettis property. This shows that Theorem 2 fails for Banach lattices with a weak order unit.

A Banach space is called *weakly compactly generated* (WCG) if it is the closed linear span of one of its weakly compact subsets. Every separable and every reflexive Banach space is WCG. It is well known [J] that every WCG space with the Grothendieck and the Dunford-Pettis property is finite dimensional.

Corollary 3. *If an infinite-dimensional Banach lattice E has the Lotz property, then E cannot be weakly compactly generated.*

Proof. Suppose E is weakly compactly generated. If E contains a copy of c_0, then by Veech's version of Sobczyk's theorem [V], this c_0 is automatically complemented. Therefore E contains no copy of c_0, and hence E has order continuous norm (see, e.g, [S, Theorem II.5.15]). In particular, closed ideals are projection bands. Since E is infinite dimensional, there is an $u > 0$ such that $\overline{E_u}$ is infinite dimensional. If every C_0-semigroup on $\overline{E_u}$ is uniformly continuous, then $\overline{E_u}$ is isomorphic to a $C(K)$ by Theorem 2 and hence contains a copy of c_0, a contradiction. So the complemented subspace $\overline{E_u}$, and hence E, admits a C_0-semigroup with unbounded generator.

The proof shows that a Lotz lattice cannot have order continuous norm. Hence by [S, Theorem II.5.14] we have

Corollary 4. *Every infinite-dimensional σ-Dedekind complete Lotz lattice contains a sublattice isomorphic to l^∞.*

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REFERENCES

[C] T. Coulhon, *Suites des opérateurs sur un espace $C(K)$ de Grothendieck*, C. R. Acad. Sci. Paris Sér. I Math. **298** (1984), 13–15.

[J] W. B. Johnson, *No infinite-dimensional P space admits a Markusevich basis*, Proc. Amer. Math. Soc. **26** (1970), 467–468.

[KR] A. Kishimoto and D. W. Robinson, *Subordinate semi-groups and order properties*, J. Austral. Math. Soc. Ser. A **31** (1981), 59–76.

[Le] D. Leung, *Uniform convergence of operators and Grothendieck spaces with the Dunford-Pettis property*, Math. Z. **197** (1988), 21–32.

[L] H. P. Lotz, *Uniform convergence of operators on L^∞ and similar spaces*, Math. Z. **190** (1985), 207–220.

[Ne] J. M. A. M. van Neerven, *Abstract multiplication semigroups* (submitted).

[N] C. P. Niculescu, *Weak compactness in Banach lattices*, J. Operator Theory **6** (1981), 217–231.

[S] H. H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, Berlin, Heidelberg, and New York, 1974.

[V] W. A. Veech, *A short proof of Sobczyk's theorem*, Proc. Amer. Math. Soc. **28** (1971), 627-628.

[W] A. W. Wickstead, *Representation and duality of multiplication operators on Archimedean Riesz spaces*, Compositio Math. **35** (1977), 225-238.

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