PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 116, Number 2, October 1992

CROSSED PRODUCTS OF SEMISIMPLE COCOMMUTATIVE HOPF ALGEBRAS

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(Communicated by Maurice Auslander)

ABSTRACT. We provide a short proof of an analog of Nagata's theorem for finite-dimensional Hopf algebras. The result, proved Hopf-algebraically by Sweedler and using group schemes by Demazure and Gabriel, says that a finite-dimensional cocommutative semisimple irreducible Hopf algebra is commutative. With mild base field assumptions such a Hopf algebra is just the dual of a p-group algebra. We give *en route* an easy proof of a version of Hochschild's theorem on semisimple restricted enveloping algebras.

Let $R\#_t H$ denote a crossed product with an invertible cocycle t, where H is a semisimple cocommutative Hopf algebra H over a perfect field. The result above is applied to show that $R\#_t H$ is semiprime if and only if R is H-semiprime. The approach relies on results on ideals of the crossed product that are stable under the action of the dual of H and the Fisher-Montgomery theorem for crossed products of finite groups.

INTRODUCTION

We begin by proving an analog of Nagata's theorem for finite-dimensional Hopf algebras. In Hopf algebraic language the result says that a finite-dimensional cocommutative semisimple irreducible Hopf algebra is commutative. With base field assumptions such a Hopf algebra is the dual of an abelian p-group algebra.

This is the finite version of a theorem of Demazure and Gabriel ("le théorème de Nagata" [DG, p. 509]). The result says that an affine algebraic connected group scheme whose rational representations are all completely reducible is a group scheme of multiplicative type. As Schneider pointed out to us, Sweedler proved this fact Hopf-algebraically [S2]. His proof relies on Hochschild's Theorem [H] and the quite technical generalization of the PBW theorem [S1] to irreducible cocommutative Hopf algebras.

In §1 we present a simplification of "Nagata's Theorem" for finite-dimensional Hopf algebras, evading use of the PBW result and the theory of group scheme

Received by the editors April 24, 1990 and, in revised form, March 18, 1991.

1991 *Mathematics Subject Classification*. Primary 16N60, 16W30, 16S40; Secondary 17B50, 14L15.

The author was partially supported by a grant from the Fund for Faculty Research and Development of the College of Liberal Arts and Sciences at De Paul University.

0002-9939/92 \$1.00 + \$.25 per page

extensions. We give *en route* an easy proof of Hochschild's theorem on semisimple restricted enveloping algebras. Much of the material in this section is not new and is due to the proofs in [S1, DG]; however this attenuated approach should be useful for algebraists interested in finite-dimensional Hopf algebras.

Let $R\#_t H$ denote a crossed product with an invertible cocycle t, where H is a semisimple cocommutative Hopf algebra H (over a perfect field). The result of §1 allows us to deduce $R\#_t H$ is semiprime if and only if R is H-semiprime in §2. The proof relies on a result concerning H^*-stable ideals of $R\#_t H$ and the Fisher-Montgomery theorem for crossed products of finite groups.

Let us now summarize some related Maschke-type results. Let H be a semisimple Hopf algebra and assume that crossed products have invertible cocycles. Cohen and Fischman [CF] asked whether a smash product $R\# H$ over a semiprime algebra R and semisimple Hopf algebra H must be semiprime. In [CM] the result follows from duality when H is the dual of a group algebra (assuming R is H-semiprime). This was generalized to crossed products with invertible cocycles over group algebra duals H in [BM2] with R semiprime. Considering inner actions [BCM, §6] showed that if R is semiprime and H is cocommutative and has an inner action on R, then $R\#_t H$ is semiprime. In [BM2] this was improved by removing the cocommutative hypothesis. A related result appears in [Ch1] where the action is X-inner and R is H-prime.

Throughout, H shall denote a finite-dimensional Hopf algebra over the field k of char $p \geq 0$, with coradical filtration denoted by $H_0 \subset H_2 \subset \cdots \subset H_n = H$, group-likes $G(H)$, primitives $P(H)$, comultiplication Δ, antipode s, and counit ε. The adjoint left action of H on itself shall be denoted $h.a = \sum h_{(1)} a s(h_{(2)})$; h, $a \in H$. We shall let A^+ denote the $A \cap \ker \varepsilon$ for all subsets $A \subset H$. $T = R\#_t H$ shall denote a (unital associative) crossed product with invertible cocycle t, as described for instance in [BCM].

The approach is fairly self-contained, with the use of material from the books [A, S3].

The author thanks S. Montgomery for her comments.

1. Nagata's theorem for finite-dimensional Hopf algebras

1.0. We shall freely use the fact [S3, 10.0.2] that

$$\Delta h - h \otimes 1 + 1 \otimes h \in H_m^+ \otimes H_m^+ \quad \text{for all } h \in H_{m+1}^+$$

when H is irreducible as a coalgebra.

Lemma [S2]. *Suppose H is an irreducible Hopf algebra. Let A be a semisimple commutative algebra that is an H-module algebra. Then the action of H on A is trivial. Thus if A is an ad_H-stable subspace of H, then A is central in H.*

Proof. We may assume that k is an algebraically closed field. Thus A is spanned by central primitive idempotents.

It is always true that $H_0 = k1$ acts trivially. Now let $h \in H_m^+$ and write $\Delta h = h \otimes 1 + 1 \otimes h + Y$ where $Y \in H_{m-1}^+ \otimes H_{m-1}^+$. It follows inductively that H_m^+ is trivial on A: First note that $h.e = h.e^2 = 2e(h.e)$; thus we have $e(h.e) = h.e = 2h.e$. Therefore $h.e = 0$.

Since $H_m = H_m^+ \oplus k$, we see that H_m is trivial on A.

To see that H centralizes A observe that

$$\begin{aligned} ha &= \sum h_{(1)} a \varepsilon(h_{(2)}) = \sum h_{(1)} a s(h_{(2)}) h_{(3)} \\ &= \sum (h_{(1)}.a) h_{(2)} = \sum \varepsilon(h_{(1)}) a h_{(2)} = ah. \quad \square \end{aligned}$$

1.1. The following consequence of [NZ] appears in [LR]. We include the proof for the reader's convenience.

Proposition 1. *Let H be a finite-dimensional semisimple Hopf algebra. Then every Hopf subalgebra is also semisimple.*

Proof. Let A be a Hopf subalgebra of H. By [NZ] H is a free left A-module, say $H = \bigoplus Ah_i$. Let e be a nonzero left integral of H. Write $e = \sum e_i h_i$, $e_i \in A$. Observe that for $a \in A$, $ae = \varepsilon(a)e = \sum \varepsilon(a) e_i h_i$; on the other hand $ae = \sum (ae_i) h_i$. We deduce by freeness that the e_i are left integrals of A. Since $\varepsilon(e) \neq 0$, we have $\varepsilon(e_i) \neq 0$, for some i and hence A is semisimple. $\square$

If H is irreducible, the fact that H is free over any Hopf subalgebra is an easier, older result of Radford [R, Corollary 2].

1.2. We point out a simple proof of a result of Hochschild.

Proposition 2. *Let $H = u(L)$ be a finite-dimensional semisimple restricted enveloping algebra. Then H is commutative.*

Proof. Without loss of generality we assume that k is algebraically closed. Let $x \in L$, and let u_x denote $u(kx + kx^p + \cdots)$, the (Hopf) subalgebra generated by x. Note that u_x is a commutative Hopf subalgebra of H. Now consider the action of u_x on the k-vector space L. As u_x is semisimple (the previous lemma), L is spanned by $\operatorname{ad} x$ eigenvectors.

Let $y \in L$ be an eigenvector for $\operatorname{ad} x$, with $[x, y] = \lambda y$, $\lambda \in k$. Thus as a Hopf algebra, u_x acts on u_y, as can be easily checked. Since u_y is semisimple, Lemma 1 shows that this action is trivial and we conclude that $\lambda = 0$. Thus x is a central element of L. $\square$

1.3. The proof below is partly due to Montgomery whose suggestion shortened our original proof, which used Hopf kernels.

Theorem 1 [DG, S1]. *Let H be a finite-dimensional semisimple cocommutative Hopf algebra.*

(a) *If H is irreducible as a coalgebra, then H is commutative and $H \otimes K \cong K[P]^*$, the dual of the group algebra of a p-group P, for some finite separable extension K of k.*

(b) *$[L]H \otimes L \cong L[P]^* \# L[G]$ with $G = G(H \otimes L)$ for some finite extension L of k and p does not divide $|G|$.*

Proof. For the first conclusion in (a) we may assume that k is algebraically closed. Since H is finite-dimensional, we can further assume that k is of positive characteristic p by Kostant's theorem.

Let A denote the largest proper $\operatorname{ad} H$-stable Hopf subalgebra of H. By induction on dimension, A is commutative; hence A is central by Lemma 1.0. We may also assume that A contains $u(P(H))$ in view of Proposition 2 (note that $u(P(H))$ embeds in H by [S3, 11.0.1]).

Let m be the largest integer with $H_m \subseteq A$ and choose $b \in H_{m+1}^+ \backslash A$. We can write

$$\Delta b = b \otimes 1 + 1 \otimes b + \sum b_i' \otimes b_i'',$$

where $\sum b_i' \otimes b_i'' \in H_m^+ \otimes H_m^+ \subseteq A^+ \otimes A^+$.

We claim that $B = A + kb$ is an ad H-stable subcoalgebra of A. Let $h \in H^+$ and write $\Delta h = h \otimes 1 + 1 \otimes h + \sum h_i' \otimes h_i''$ where $\sum h_i' \otimes h_i'' \in H^+ \otimes H^+$. Let us observe that $h.b$ is primitive: Observe that

$$\begin{aligned}\Delta(h.b) &= \sum h_1.b_1 \otimes h_2.b_2 \quad (\text{use } H \text{ cocommutative})\\ &= h.b \otimes 1 + 1 \otimes h.b + b \otimes \varepsilon(h) + \varepsilon(h) \otimes h.b + \sum h_j'.b_i' \otimes h_j''.b_i''\\ &= h.b \otimes 1 + 1 \otimes h.b,\end{aligned}$$

where the last three terms in the second line are zero because H is trivial on A and $h, h_j', h_j'' \in H^+$. Thus $h.b \in P(H) \subset A$. It follows immediately that B is H-stable. Since A is central, the Hopf subalgebra generated by B is commutative; it is also H-stable. This contradicts the maximality of A.

The second statement in (a) is a slight amplification of Harrison's theorem (cf. [L]) where k is not assumed to be algebraically closed. One may consult [BeC] for a proof of this fact that deals with the field extension issue (noting that H is a separable algebra). The group $P = G(H^*)$ is an abelian p-group, since by hypothesis, H^* is a commutative local group algebra.

(b) H is pointed after finite base extension by [Ch2, 3.1]. Now use the well-known [S3, 8.1.5] for the decomposition. Finally, Mashcke's Theorem implies that $p \nmid |G|$. □

2. Crossed products

2.0. An algebra R is said to be H-prime if the product of nonzero H-stable ideals is nonzero. R is *H-semiprime* if zero is the only nilpotent H-stable ideal, or equivalently, zero is the intersection of H-prime ideals. The main goal of this section is to show how the decomposition above can be used to deduce the following

Theorem 2. *Let $R \#_t H$ be a crossed product with an invertible cocycle t. Assume that H is finite-dimensional cocommutative semisimple and k is a perfect field. Then R is H-semiprime if and only if $R \#_t H$ is semiprime.*

We do not know whether the field assumption is necessary.

2.1. Let us begin by mentioning some easy consequences of Theorem 1.4. Suppose H is semisimple and cocommutative and k is big enough so that $H = k[P]^* \# k[G]$ as in Theorem 1(b). It is apparent that some results known for smash products of finite group algebras (see, e.g., [P]) and their duals (e.g., [BeC, CM]) quickly yield results for actions of H on R and smash products $R \# H$. For example by [CM] if R is semiprime, so is $R \# k[P]^* = R \# U$. Thus by the Fisher-Montgomery Theorem $R \# H = (R \# U) \# k[G]$ is semiprime. In addition using [BeM, 2.4] we find that R^H is semiprime too.

Other results may be obtained in this manner. For example using [BeC] for $k[P]^*$ actions and [M, Theorem 6.5] for group actions, it follows easily that

R satisfies a polynomial identity iff R^H does. Other generalizations of statements known for group actions and gradings are not as apparent, for example, integrality questions [Q].

2.2. The following proposition extends the smash product result [Ch2, Theorem 1.4] to crossed products, when H is irreducible.

We use a map defined in [BM1, p. 156] to obtain an action of H^* on $T = R\#_t H$ as follows. Let $h \in H$ and $u \in H^*$. The map $\lambda\colon H^* \to \mathrm{End}_k(T)$ is an algebra map making T a left H^*-module algebra given by

$$\lambda(u)(r\# h) = r\#(u \rightharpoonup h),$$

where the left action H^* on H is given by $u \rightharpoonup h = \sum\langle u, h_2\rangle h_1$. It is easy to check that the elements act as right $R = R\#_t 1$-module maps.

Doi pointed out that the following proposition holds more generally in the context of Hopf-Galois extensions [DT, Theorem 2.11(a)]. We state a direct consequence for crossed products:

Proposition 3. *Let $T = R\#_t H$ be a crossed product where H is a finite-dimensional Hopf algebra. If I is a H^*-stable ideal of T, then $I = (I \cap R)T$.*

Let us supply an easy proof when H is *irreducible*: Define a basis for H as follows. For all $i > 0$, let $C_0 = k1$ and C_i be a k-complement for H_{i-1}^+ in H_i^+. Thus $H_r = k \oplus C_1 \oplus C_2 \oplus \cdots \oplus C_r$ as k-vector spaces. For each i, let $\{h_{ij}\}$ be a basis for C_i. Note that $\Delta h_{ij} - h_{ij} \otimes 1 - 1 \otimes h_{ij} \in H_{i-1}^+$.

Further let f_{ij} denote elements of a basis of H^*, dual to the h_{ij}. Observe that $f_{ij} \rightharpoonup h_{lk} = f_{ij}(h_{kl}) = 1$ if $i = l$ and $j = k$, and zero otherwise.

Let $\alpha = \sum a_{ij}h_{ij} \in I$. By [Ch1, 1.4(iii), 1.5(b)] we may write coefficients on the right, i.e., $\alpha = \sum(1\# h_{ij})(b_{ij}\# 1)$ for some $b_{ij} \in R$. Now we easily compute that $f_{ij} \rightharpoonup \alpha = b_{ij} \in I \cap R$ for all i, j.

Hence we have shown that $T(I\cap R)$ contains I. Finally $T(I\cap R) = (I\cap R)T \subseteq I$, again using [Ch1, 1.4(iii), 1.5].

2.3. **Stable ideals.** Let T denote $R\#_t H$, a crossed product with an invertible cocycle t, where H is a Hopf algebra with invertible antipode (for example H finite-dimensional or cocommutative). We shall make use of the *twisted module condition* ([BCM, p. 691] or see [Ch2]), which can now be written as

$$h.(l.r) = \sum t(h_{(1)}, l_{(1)})(h_{(2)}l_{(2)}.r)t^{-1}(h_{(3)}, l_{(3)}) \tag{1}$$

or

$$(hl).r = \sum t^{-1}(h_{(1)}, l_{(1)})(h_{(2)}.(l_{(2)}.r))t(h_{(3)}, l_{(3)}) \tag{2}$$

for all $h, l \in H$ and $r \in R$.

Let A be an ideal of R, and define

$$(A : H) = \{r \in R \mid h.r \in A, \text{ all } h \in H\}.$$

It follows from the twisted module condition that $(A : H)$ is an H-stable ideal of R, where A is an ideal of R [Ch1]. Thus $(A : H)$ is the largest H-stable ideal contained in A.

In case $H = k[G]$, we are in the situation of ordinary group crossed products as described in, e.g., [P]. The invertibility of t implies that the image of t is contained in the units of R, and (using measuring and the twisted module condition) that each $g \in G$ acts as an automorphism of R. Further, as usual, the weak action of $k[G]$ induces a permutation action of G on the set of ideals of R. Thus $(A : H)$ is the intersection of the orbit of $A\cdot$ under the group G.

Lemma 2. *Assume further that H splits as a smash product $U \# V$ where U and V are Hopf subalgebras. Let Q be an ideal of R; then*

(a) $(Q : H) = ((Q : U) : V)$.

(b) *If $U = k[P]^*$ and $V = k[G]$ as in Theorem* 1(b) (*with $k = L$*), *then* (i) *$g.(Q : U)$ is a U-stable ideal of R for every $g \in G$; in particular, if Q' is a U-stable ideal of R, then so is $g.Q'$ for every $g \in G$;* (ii) *if Q is H-semiprime, it is U-semiprime.*

Proof. Write $h = u \# v \in H$, and let $r \in R$. The twisted module condition (2) above yields

$$h.r = (u \# v).r = \sum t^{-1}(u_{(1)}, v_{(1)})(u_{(2)}.(v_{(2)}.r))t(u_{(3)}, v_{(3)}) \in RU.(V.r)R.$$

Thus if $r \in ((Q : U) : V)$ then $h.r \in (Q : H)$. The other inclusion follows similarly from condition (1). This finishes (a).

Let $g \in G$ and suppose $r \in Q'$, a U-stable ideal. As $\Delta g = g \otimes g$, we have

$$\begin{aligned} u.(g.r) &= \sum t^{-1}(u_{(1)}, g)((u \# g).r)t(u_{(2)}, g) \\ &= \sum t^{-1}(u_{(1)}, g)((1 \# g).((g^{-1}.u_{(2)} \# 1).r))t(u_{(3)}, g) \\ &\in Rg.(U.r)R \subset R(g.Q')R \subset (g.Q'). \end{aligned}$$

Thus (i) follows.

Since H is finite-dimensional, by Zorn's Lemma, there exists an ideal P maximal with respect to $(P : H) = 0$. P is easily seen to be semiprime, so that now $P' = (P : U)$ is U-semiprime. By (a), $(P' : k[G]) = (P : H) = 0$. Thus, it suffices to show that $g.P'$ is U-semiprime for all $g \in G$. Accordingly, suppose A is a U-stable ideal of R with $A^2 \subset g.P'$. Then

$$(g^{-1}.A)^2 = g^{-1}.(A^2) \subset P',$$

which implies, using (i), that $g^{-1}.A \subset P'$ and thus $A \subset g.P'$. This completes the proof of (ii). □

Proof of Theorem 2. Since k is perfect we may replace k with a finite Galois extension so that H splits as $k[P]^* \# k[G]$. (We can get by with a finite extension to make H pointed by [Ch2, 3.1].) This can be done without loss of generality since, using the Galois group action, standard arguments show that being H-semiprime or semiprime is unaffected by such an extension.

Notice that T is a crossed product of the group G over $R \#_t U$, since T is graded by G and each component $R \#_t (U \# g)$ contains a unit. Thus by the Fisher-Montgomery Theorem (see [M] or [P]), if R is H-semiprime, it suffices to show that $R \#_t U$ is semiprime.

By Lemma 2(b)(ii), R is U-semiprime. Using Proposition 3.2, it follows that $R \#_t U$ is $U^* = k[P]$-semiprime, and hence semiprime.

Conversely if $R \#_t H$ is semiprime, then any H-stable ideal of R generates an ideal $AT = TA$ of T [Ch1, 1.5]. Plainly, A cannot be nilpotent. □

If H is irreducible, Theorem 1(a) applies, and no field assumption is necessary. The cocycle t is necessarily invertible by [Ch1, 1.4].

Corollary 1. *Let $R\#_t H$ be a crossed product where H is a finite-dimensional semisimple irreducible cocommutative Hopf algebra. Then $R\#_t H$ is semiprime iff R is H-semiprime.*

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