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ON THE THOM SPECTRA OVER $\Omega(\mathrm{SU}(n)/\mathrm{SO}(n))$ AND MAHOWALD'S X_k SPECTRA

DUNG YUNG YAN

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ABSTRACT. The Thom spectra $M(n)$ ($2^k \leq n \leq 2^{k+1} - 1$) induced from $\Omega(\mathrm{SU}(n)/\mathrm{SO}(n)) \to BO$ is a wedge of suspensions of Mahowald's X_k spectra that is induced from $\Omega J_{2^k-1}S^2 \to \Omega^2 S^3 \to BO$, where J_i is the ith stage of the James construction.

Given a connected H-space L and an H-map $f: L \to BO$, the resulting Thom spectrum $T(f)$ is a ring spectrum with a two-sided unit [1, 5]. If L and BO admit associating homotopies compatible under f, then $T(f)$ is an associative ring spectrum, and if L has higher multiplicative structure compatible with BO under f, then $T(f)$ has analogous structure in the multiplication of $T(f)$.

$T(f)$ is (-1)-connected and $\pi_0(T(f))$ is either $\mathbb{Z}$ or $\mathbb{Z}/2$. If f is nonorientable, i.e., $f^*(w_1) \neq 0$, then $\pi_0(T(f)) = \mathbb{Z}/2$. Otherwise $\pi_0(T(f)) = \mathbb{Z}$.

For our purpose a ring spectrum is a spectrum with a multiplication that is associative and with a two-sided unit, but not necessarily commutative. Furthermore "a ring map" will mean "a map between two ring spectra that is multiplicative and carries the unit," otherwise "a map" even between two ring spectra is not necessarily multiplicative.

Let $\eta: S^1 \to BO$ represent the generator of $\pi_1(BO) = \mathbb{Z}/2$. Since BO is a double loop space there is an induced map $\gamma: \Omega^2 S^3 \to BO$. Then one takes the composite map $\Omega J_{2^k-1}S^2 \to \Omega^2 S^3 \to BO$, where J_i is the ith stage of the James construction. These maps result in Thom spectra which will be denoted by X_k due to Mahowald [1, 3, 4, 5]. Also let

$$\Omega(\mathrm{SU}(n)/\mathrm{SO}(n)) \to \Omega(\mathrm{SU}/\mathrm{SO}) = BO$$

be the usual inclusion map. These maps yield the Thom spectra $M(n)$. In the present note, we intend to show $M(n)$ is a wedge of suspensions of X_k for $2^k \leq n \leq 2^{k+1} - 1$, which reflects a well-known splitting in the complex case originally due to Ravenel, elucidated by Hopkins [2]. An application of this paper will appear in [8].

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Theorem 1. *$M(n)$ is a wedge of suspensions of X_k for $2^k \leq n \leq 2^{k+1}-1$.*

From now on all unstated coefficient groups are $\mathbb{Z}/2$. A is the mod 2 dual Steenrod algebra, and let $\xi_i \in A_{2^i-1}$ be the generator defined by Milnor. Then as an algebra $A = \mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \dots]$, and the coproduct is determined by $\Delta\xi_k = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i$. The Thom spectra $M(n)$ and X_k are ring spectra, so one will know that $H_*(M(n))$, $H_*(X_k)$ are (left) A comodule algebras in the following two propositions.

Proposition 2 [1, 5]. *As a subcomodule algebra of A, it follows that $H_*(X_k) \cong \mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_k]$.*

There is a well-known inclusion $\mathbb{R}P^{n-1} \to \Omega(\mathrm{SU}(n)/\mathrm{SO}(n))$ such that

$$H_*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n))) \cong \mathbb{Z}/2[c_1, c_2, \dots, c_{n-1}], \qquad |c_i| = i,$$

c_i is induced from the $\mathbb{R}P^i$, $1 \leq i \leq n-1$, and

$$H_*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n))) \to H_*(BO)$$

is injective. The Thom isomorphism yields

$$H_*(M(n)) \cong \mathbb{Z}/2[\bar{c}_1, \bar{c}_2, \dots, \bar{c}_{n-1}].$$

Since $H_*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n)))$ injects in $H_*(BO)$, $H_*(M(n)) \to H_*(MO)$ is an injection. It is an isomorphism for $* \leq n-1$. To get the mod two homology splitting of $M(n)$ over the dual Steenrod algebra, recall the splitting of $H_*(MO)$ over the dual Steenrod algebra

$$H_*(MO) \cong \mathbb{Z}/2[\xi_1, \xi_2, \dots] \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\bar{\beta}_i / i \neq 2^l - 1], \qquad |\bar{\beta}_i| = i,$$

and $\bar{\beta}_i$ has the trivial coaction over the dual Steenrod algebra.

Note. We actually choose the above splitting from the following isomorphism (as comodule algebras)

$$\pi_*(MO) \otimes_{Z/2} H_*(M(\gamma)) \xrightarrow{h_* \otimes \gamma_*} H_*(MO) \otimes_{\mathbb{Z}/2} H_*(MO) \xrightarrow{m} H_*(MO),$$

where $h_*: \pi_*(MO) \to H_*(MO)$ is the mod 2 Hurewicz map, $M(\gamma)$ is the Thom spectrum from $\gamma: \Omega^2 S^3 \to BO$, $\gamma_*: H_*(M(\gamma)) \to H_*(MO)$ is the map induced from $\gamma: \Omega^2 S^3 \to BO$, and m is the multiplication map.

Proposition 3. *$H_*(M(n)) \cong \mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_p] \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\bar{y}_i / i \neq 2^l - 1]$, $2^p \leq n$, $i \leq n-1$, $|\bar{y}_i| = i$, and $\bar{y}_i$ has the trivial coaction over A.*

Proof. Since $H_*(M(n)) \to H_*(MO)$ is injective and isomorphic for $* \leq n-1$, it follows that $H_*(M(n))$ is a left $\mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_p]$ comodule algebra from the splitting of $H_*(MO)$. Moreover it maps onto $\mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_p]$. Hence, by a theorem of Milnor and Moore [6],

$$H_*(M(n)) \cong \mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_p] \otimes_{\mathbb{Z}/2} \mathbf{C}$$

as left $\mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_p]$ comodule algebras, where

$$\mathbf{C} = \mathbb{Z}/2 \,\square_{\mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_p]} H_*(M(n)).$$

An easy counting argument shows that $\mathbf{C}$ must have the indicated form. □

Before going to the Thom splitting results, we would like to prove the following property.

Proposition 4. *X_k and $M(n)$ are noncommutative ring spectra for $k \geq 1$ and $n \geq 2$ respectively.*

The proof uses the following result.

Theorem 5 [2, 2.2.1]. *Let E be a commutative ring spectrum. If $\pi_*(E)$ contains an invertible element of order two then E is weakly equivalent to a wedge of suspensions of Eilenberg-Mac Lane spectra.*

Also recall that X_k is taken to be the Thom spectrum induced from

$$\Omega J_{2^k-1}S^2 \longrightarrow \Omega^2 S^3 \longrightarrow BO,$$

and we know that $\Omega^2 S^3 \to BO$ induces the mod 2 Eilenberg-Mac Lane spectrum $H(\mathbb{Z}/2)$ [1, 4, 5], hence one has the ring map $X_k \to H(\mathbb{Z}/2)$ that induces the natural embedding of mod 2 homology.

Proof of Proposition 4. Since the pull back of the first Stiefel-Whitney class is nonzero for $\Omega J_{2^k-1}S^2 \to BO$ $(k \geq 1)$, then $\pi_0(X_k) = \mathbb{Z}/2$, i.e., the unit is the invertible element of order two. If X_k were a commutative ring spectrum, then X_k would be a wedge of suspensions of Eilenberg-Mac Lane spectra. So if the cohomology of X_k was a free module over the Steenrod algebra, then one would have the ring map $X_k \to H(\mathbb{Z}/2)$, which would induce cohomology injective, a contradiction. An analogous argument holds for $M(n)$. □

Theorem 6. *$M(2^{k+1}-1)$ is a wedge of suspensions of X_k.*

Lemma 7. *The usual inclusion $\mathrm{SU}(n)/\mathrm{SO}(n) \to \mathrm{SU}(n+1)/\mathrm{SO}(n+1)$ induces an isomorphism on π_* for $* \leq n-1$, an epimorphism for $* = n$ if n is odd and $n \geq 3$, and an isomorphism on π_* for $* \leq n-2$, an epimorphism for $* = n-1$ if $n \geq 3$.*

To prove the lemma, we quote the classical Zeeman's comparison theorem [9]. A homomorphism between two spectral sequences (both are 1st quadrant spectral sequences). That is to say,

$$\begin{array}{ll}
f^r_{p,q}: E^r_{p,q} \to \bar{E}^r_{p,q} & (r = 2, 3, \ldots, \infty \text{ and } -\infty < p,\ q < \infty), \\
f^A_n: A_n \to \bar{A}_n & (n = 0, 1, \ldots), \\
f^B_p: B_p \to \bar{B}_p & (p = 0, 1, \ldots), \\
f^C_q: C_q \to \bar{C}_q & (q = 0, 1, \ldots),
\end{array}$$

where $A_n = \bigoplus_{p+q=n} E^\infty_{p,q}$, $B_p = E^2_{p,0}$, $C_q = E^2_{0,q}$, and the same definitions as for $\bar{A}_n$, $\bar{B}_p$, $\bar{C}_q$. Furthermore the spectral sequences satisfy the following commutative diagram with rows exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E^2_{p,0} \otimes E^2_{0,q} & \longrightarrow & E^2_{p,q} & \longrightarrow & \mathrm{Tor}(E^2_{p-1,0}, E^2_{0,q}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bar{E}^2_{p,0} \otimes \bar{E}^2_{0,q} & \longrightarrow & \bar{E}^2_{p,q} & \longrightarrow & \mathrm{Tor}(\bar{E}^2_{p-1,0}, \bar{E}^2_{0,q}) & \longrightarrow & 0
\end{array}$$

We say that f^A is an isomorphism up to dimension N if f^A_n is an isomorphism for each n $(0 \leq n \leq N)$. Then

Theorem 8 [9]. *If f^A, f^C are isomorphisms up to dimensions N and Q respectively, then f^B is an isomorphism up to dimension $P = \min(N, Q)$. If, further, f^A_{P+1} is onto, then f^B_{P+1} is onto.*

Proof of Lemma 7. Consider the following diagram with each row and each column a fibration:

$$\begin{tikzcd}
\mathrm{SO}(n) \arrow[r] \arrow[d] & \mathrm{SO}(n+1) \arrow[r] \arrow[d] & S^n \\
\mathrm{SU}(n) \arrow[r] \arrow[d] & \mathrm{SU}(n+1) \arrow[r] \arrow[d] & S^{2n+1} \\
\mathrm{SU}(n)/\mathrm{SO}(n) \arrow[r] & \mathrm{SU}(n+1)/\mathrm{SO}(n+1) &
\end{tikzcd}$$

It follows from the fibrations that

$$H_*(\mathrm{SO}(n), \mathbb{Z}) \longrightarrow H_*(\mathrm{SO}(n+1), \mathbb{Z})$$

is an isomorphism for $* \leq n-2$, a surjection for $* = n-1$, and

$$H_*(\mathrm{SU}(n), \mathbb{Z}) \longrightarrow H_*(\mathrm{SU}(n+1), \mathbb{Z})$$

is an isomorphism for $* \leq 2n-1$. In order to let Zeeman's comparison work, we use the $\mathbb{Z}/p$ coefficient for each prime p in the Serre spectral sequences. Then to prove the second statement is easy. Since $n-2 < n-1 < 2n-1$ for $n \geq 3$, from Theorem 8 the map $\mathrm{SU}(n)/\mathrm{SO}(n) \to \mathrm{SU}(n+1)/\mathrm{SO}(n+1)$ induces an isomorphism on $H_*(\ \ , \mathbb{Z}/p)$ for $* \leq n-2$ and a surjection on $H_{n-1}(\ \ , \mathbb{Z}/p)$ for each prime p. Since the integral homology of $\mathrm{SU}(n)/\mathrm{SO}(n)$ and $\mathrm{SU}(n+1)/\mathrm{SO}(n+1)$ are of finite type, a trivial application of the mapping cylinder of

$$\mathrm{SU}(n)/\mathrm{SO}(n) \to \mathrm{SU}(n+1)/\mathrm{SO}(n+1),$$

one can show $\mathrm{SU}(n)/\mathrm{SO}(n) \to \mathrm{SU}(n+1)/\mathrm{SO}(n+1)$ induces an isomorphism on $H_*(\ \ , \mathbb{Z})$ for $* \leq n-2$ and a surjection for $* = n-1$. Hence by Hurewicz's theorem the map induces an isomorphism on π_* for $* \leq n-2$ and a surjection for $* = n-1$. This proves the second statement. In the first statement, if one can show $H_{n-1}(\mathrm{SO}(n), \mathbb{Z}) \to H_{n-1}(\mathrm{SO}(n+1), \mathbb{Z})$ is an isomorphism when n is odd and $n \geq 3$, i.e., $H_*(\mathrm{SO}(n), \mathbb{Z}) \to H_*(\mathrm{SO}(n+1), \mathbb{Z})$ is an isomorphism for $* \leq n-1$, then by Theorem 8 again the map $\mathrm{SU}(n)/\mathrm{SO}(n) \to \mathrm{SU}(n+1)/\mathrm{SO}(n+1)$ induces an isomorphism on $H_*(\ \ , \mathbb{Z}/p)$ for $* \leq n-1$ and a surjection on $H_n(\ \ , \mathbb{Z}/p)$ for each prime p since $n-1 < n < 2n-1$. Therefore, with an argument analogous to the one above we can prove the first statement. So it remains to prove that $H_{n-1}(\mathrm{SO}(n), \mathbb{Z}) \to H_{n-1}(\mathrm{SO}(n+1), \mathbb{Z})$ is an isomorphism when n is odd. In the following paragraph we will use the integral coefficients.

By the Whitehead theorem and the long exact sequence of the pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ one has

$$H_n(\mathrm{SO}(n+1), \mathrm{SO}(n)) \xrightarrow{\partial} H_{n-1}(\mathrm{SO}(n)) \longrightarrow H_{n-1}(\mathrm{SO}(n+1)) \longrightarrow 0.$$

Let $\mathrm{SO}^n(n+1) = K^n \cup \mathrm{SO}(n)$ where K^n is the n-skeleton of $\mathrm{SO}(n+1)$. Then for the triple $(\mathrm{SO}(n+1), \mathrm{SO}^n(n+1), \mathrm{SO}(n))$ we also have a long exact sequence

$$\begin{aligned} &\longrightarrow H_n(\mathrm{SO}^n(n+1), \mathrm{SO}(n)) \longrightarrow H_n(\mathrm{SO}(n+1), \mathrm{SO}(n)) \\ &\longrightarrow H_n(\mathrm{SO}(n+1), \mathrm{SO}^n(n+1)) \longrightarrow . \end{aligned}$$

From the cellular homology, $H_n(\mathrm{SO}(n+1), \mathrm{SO}^n(n+1)) = 0$. So the first map is surjective. Again from the cellular homology, the well-known CW-decomposition of $\mathrm{SO}(n+1) = \mathrm{O}(n+1)/\mathrm{O}(1)$ [7], and the fact that n is odd ($n \geq 3$), one has

$$\begin{tikzcd} \mathbb{Z} \cong H_n(\mathbb{R}P^n_+, \mathbb{R}P^{n-1}_+) \arrow[r, "\cong"] \arrow[dr] & H_n(\mathrm{SO}^n(n+1), \mathrm{SO}(n)) \cong \mathbb{Z} \arrow[d, "\text{onto}"] \\ & H_n(\mathrm{SO}(n+1), \mathrm{SO}(n)) \end{tikzcd}$$

Hence

$$\begin{tikzcd} H_n(\mathbb{R}P^n_+, \mathbb{R}P^{n-1}_+) \arrow[r, "\text{onto}"] \arrow[d] & H_n(\mathrm{SO}(n+1), \mathrm{SO}(n)) \arrow[d, "\partial"] \\ H_{n-1}(\mathbb{R}P^{n-1}_+) \arrow[r] & H_{n-1}(\mathrm{SO}(n)) \end{tikzcd}$$

But $H_{n-1}(\mathbb{R}P^{n-1}_+) = 0$ since n is odd and $n \geq 3$. So $\partial = 0$, therefore

$$H_{n-1}(\mathrm{SO}(n)) \xrightarrow{\cong} H_{n-1}(\mathrm{SO}(n+1)).$$

This completes the proof. $\square$

Lemma 9. *There is a ring map of ring spectra* $f\colon X_k \to M(2^{k+1}-1)$ *such that* $H_*(X_k)$ *is isomorphic onto* $\mathbb{Z}/2[\xi_1, \xi_2, \dots \xi_k]$ *of*

$$H_*(M(2^{k+1}-1)) \cong \mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_k] \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\bar{y}_i \mid i \neq 2^l - 1]$$

under f_*, *that is,* f_* *projects the mod* 2 *homology of* X_k *onto a wedge summand of the mod* 2 *homology of* $M(2^{k+1}-1)$.

Proof. Using Lemma 7, it is easy to see $\mathrm{SU}(2^{k+1}-1)/\mathrm{SO}(2^{k+1}-1) \to \mathrm{SU}/\mathrm{SO}$ induces an isomorphism on π_* for $* \leq 2^{k+1}-2$ and an epimorphism for $* = 2^{k+1}-1$ when $k \geq 1$. It is known that the CW-complex $J_{2^k-1}S^2$ has dimension $2^{k+1}-2$, hence we have the following lifting (up to homotopy).

$$\begin{tikzcd} & \mathrm{SU}(2^{k+1}-1)/\mathrm{SO}(2^{k+1}-1) \arrow[d] \\ J_{2^k-1}S^2 \arrow[ur, "f"] \arrow[r] & \mathrm{SU}/\mathrm{SO} \end{tikzcd}$$

Looping the diagram and then Thomifying the resulting diagram, one has

$$\begin{tikzcd} & M(2^{k+1}-1) \arrow[d] \\ X_k \arrow[ur, "f"] \arrow[r] & MO \end{tikzcd}$$

This ring map $f: X_k \to M(2^{k+1}-1)$ is the desired one. Since each map in the above diagram is a ring map and $H_*(X_k) \to H_*(MO)$, $H_*(M(2^{k+1}-1)) \to H_*(MO)$ are injective, it follows that f_* maps $H_*(X_k)$ onto $\mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_k]$. This completes the proof. □

Proof of Theorem 6. Since

$$\pi_0(M(2^{k+1}-1)) = \pi_0(MO) = \mathbb{Z}/2,$$

not only are $M(2^{k+1}-1)$ and MO 2-local but also $\pi_*(M(2^{k+1}-1))$, $\pi_*(MO)$ are of characteristic 2. And the natural ring map $M(2^{k+1}-1) \to MO$ induces an isomorphism on $H_*(\ \ , \mathbb{Z}/2)$ for $* \leq 2^{k+1}-2$. So $\pi_*(M(2^{k+1}-1)) \to \pi_*(MO)$ is an epimorphism for $* \leq 2^{k+1}-2$. Therefore, in the splitting

$$H_*(M(2^{k+1}-1)) \cong \mathbb{Z}/2[\xi_1, \xi_2, \dots, \xi_k] \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\bar{y}_2, \bar{y}_4, \dots, \bar{y}_{2^{k+1}-2}],$$

each polynomial generator in $\mathbb{Z}/2[\bar{y}_2, \bar{y}_4, \dots, \bar{y}_{2^{k+1}-2}]$ is a mod 2 Hurewicz image. Finally, construct ring spectra $L_{2^{k+1}-1}$ out of suitable wedges of spheres, satisfying

$$H_*(L_{2^{k+1}-1}) \cong \mathbb{Z}/2[\bar{y}_2, \bar{y}_4, \dots, \bar{y}_{2^{k+1}-2}].$$

Using Lemma 9, one has the map

$$X_k \wedge L_{2^{k+1}-1} \longrightarrow M(2^{k+1}-1) \wedge M(2^{k+1}-1) \longrightarrow M(2^{k+1}-1),$$

where the last map is the multiplication map. By the construction, it follows that the above map induces isomorphism on mod 2 homology, hence, by the Hurewicz theorem, the map is an equivalence since $X_k \wedge L_{2^{k+1}-1}$, $M(2^{k+1}-1)$ are 2-local. This completes the proof. □

Proof of Theorem 1. Again, construct a ring spectra L_n out of suitable wedges of spheres, satisfying

$$H_*(L_n) \cong \mathbb{Z}/2[\bar{y}_i \mid i \neq 2^l - 1,\ 1 \leq i \leq n-1], \qquad 2^k \leq n \leq 2^{k+1}-1.$$

Obviously, we have a map $g_n: L_{2^{k+1}-1} \to L_n$. Then using Theorem 6, one yields a map

$$M(n) \longrightarrow M(2^{k+1}-1) \longrightarrow X_k \wedge L_{2^{k+1}-1} \xrightarrow{1 \wedge g_n} X_k \wedge L_n,$$

the first map is from the natural inclusion

$$\Omega(\mathrm{SU}(n)/\mathrm{SO}(n)) \longrightarrow \Omega(\mathrm{SU}(2^{k+1}-1)/\mathrm{SO}(2^{k+1}-1)).$$

It is trivial to check the mod 2 homology is an isomorphism, hence an equivalence. □

We finish the paper by pointing out a remark.

Remark 10. Under the splitting of Theorem 6, i.e., $M(2^{k+1}-1)$ is a wedge of suspensions of X_k, one has $X_k \xrightarrow{f} M(2^{k+1}-1) \xrightarrow{P} X_k$ and $Pf \simeq 1$, where f is the ring map in Lemma 9. Then

(a) P carries the unit.

(b) The following diagram commutes

$$\begin{array}{ccccc} X_k & \xrightarrow{f} & M(2^{k+1}-1) & \xrightarrow{P} & X_k \\ \downarrow & & \downarrow & & \downarrow \\ H(\mathbb{Z}/2) & \longrightarrow & MO & \longrightarrow & H(\mathbb{Z}/2) \end{array}$$

where $MO \longrightarrow H(\mathbb{Z}/2)$ is the Thom class.

(c) $P_*: H_*(M(2^{k+1}-1)) \to H_*(X_k)$ is a ring map.

Proof. (a) is obvious. For (b), the first commutative diagram is induced from

$$\begin{array}{ccc} \Omega J_{2^k-1}S^2 & \longrightarrow & \Omega(\mathrm{SU}(2^{k+1}-1)/\mathrm{SO}(2^{k+1}-1)) \\ \downarrow & & \downarrow \\ \Omega^2 S^3 & \longrightarrow & BO \end{array}$$

This diagram is actually induced from the diagram in the proof of Lemma 9. The second commutative diagram is due to

$$\begin{aligned} H^\circ(M(2^{k+1}-1)) &\cong [M(2^{k+1}-1), H(\mathbb{Z}/2)]_0 \\ &\cong \mathrm{Hom}^\circ_{\mathbb{Z}/2}(H_*(M(2^{k+1}-1), \mathbb{Z}/2), \mathbb{Z}/2) \cong \mathbb{Z}/2, \end{aligned}$$

and both maps from $M(2^{k+1}-1)$ to $H(\mathbb{Z}/2)$ are essential maps. To prove (c), from (b), one knows the map

$$H_*(M(2^{k+1}-1)) \xrightarrow{P_*} H_*(X_k) \longrightarrow H_*(H(\mathbb{Z}/2))$$

is a ring map. But the second map is the natural embedding. So P_* is a ring map. □

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Department of Mathematics, GN-50, University of Wasington, Seattle, Washington 98195

Current address: Department of Mathematics, John Hopkins University, Baltimore, Maryland 21218