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THE HAUSDORFF DIMENSION OF ELLIPTIC MEASURE— A COUNTEREXAMPLE TO THE OKSENDAHL CONJECTURE IN $\mathbb{R}^2$

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ABSTRACT. Two counterexamples to the Oksendahl conjecture in $\mathbb{R}^2$ for elliptic measure are constructed. It is shown that there exists a strictly elliptic divergence form operator in a specially constructed quasi-disk such that the associated elliptic measure has as its support a set of Hausdorff dimension arbitrarily close to 2. The method is the construction of a quasi-conformal map from a quasi-disk whose boundary has high Hausdorff dimension to the unit disk. The L-operator is the pull-back of Δ on the unit disk.

The purpose of this paper is to construct a counterexample to the Oksendahl conjecture in $\mathbb{R}^2$ for L-harmonic measure, where L is a strictly elliptic second-order operator in divergence form. In 1985 Jones and Wolff [5] proved that harmonic measure has support of Hausdorff dimension 1 on any set in $\mathbb{R}^2$. This paper shows their result does not extend to elliptic measure, in fact that for any $\varepsilon > 0$ one can construct an operator whose associated measure has support of Hausdorff dimension $2 - \varepsilon$.

The construction is due to a suggestion of Thomas Wolff. The idea is to build a quasi-conformal map from a quasi-disk whose boundary has given Hausdorff dimension to the unit disk D so that the trace of the quasi-conformal map forms a measure supported on the entire boundary. One then looks at the L-operator which is the pull-back of Δ on D to the quasi-disk; the associated elliptic measure will be supported on the entire quasi-circle.

To demonstrate the method more easily we first do the construction on the snowflake domain. The idea of iterating piecewise linear maps is due to Gehring and Väisäliä [3]. One first maps the snowflake domain onto a triangle as follows: the basic map is the piecewise linear map which takes each external triangle in the nth stage construction of the snowflake domain into S^{n-1} (see Figures 1 and 2 on the next page). Then $S^{n-1} \to S^{n-2}$ by the same basic map, etc.

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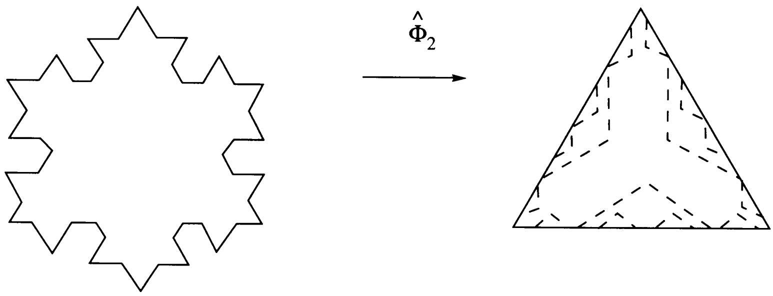

FIGURE 1

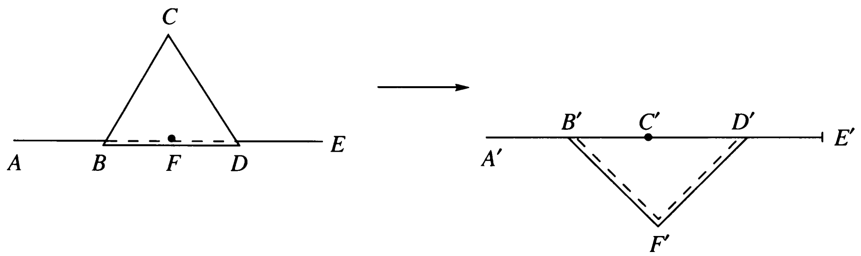

FIGURE 2

The boundary triangles T^n_{k+j} in Figure 4 are mapped to smaller similar triangles in the ratio of $3^{-n} : 4^{-n}$ so that the region near the boundary of S^n is undistorted. All the distortion occurs away from the boundary (inside the triangle $T^{n-1}_{i_k}$ in Figure 4) except at vertex points. An elementary computation shows it is possible to have the distortion at the nth stage occur inside the region which is left undistorted by previous maps and outside the area affected by future maps, with one exception which will be described below.

To make room for triangle BCD (Figure 2) the area inside BDH (Figure 3) is then compressed into region $B'F'D'H'$ (Figure 3) also by piecewise linear maps on triangles (see Figure 5).

Assume $\widehat{\Phi}_{n-1}: S^{n-1} \to S^{n-2}$ has been constructed. Define $\widehat{\Phi}_n: S^n \to S^{n-1}$ as follows: On S^n = generation n of the construction of the snowflake domain form the set of interior boundary triangles, where each triangle has base lying on a side of ∂S^n. These are isosceles triangles with base angle $\alpha < \pi/6$ and height $3^{-n}/2\tan\alpha$. Call the set $A = \{T^n_j\}_{j=1}^{3\cdot(3\cdot 4^n)}$, so that $\partial S^n \cap \{\bigcup_{j=1}^{3\cdot(3\cdot 4^n)} T^n_j\} = \partial S^n$ and $\operatorname{int} T^n_j \cap \operatorname{int} T^n_k = \varnothing$ for $j \neq k$.

Now define a piecewise linear map $\widehat{\Phi}_n$ which maps T^n_k, T^n_{k+1}, T^n_{k+2}, and T^n_{k+3} into $T^{n-1}_{i_k}$ (see Figure 4), where $\widehat{\Phi}_n(T^n_{k+j}) = T^{n'}_{k+j}$. $T^{n'}_{k+j}$ is similar to T^n_{k+j} with sides in the ratio $4^{-1} : 3^{-1}$. ($W = ABCDEF$ and $T^{n-1}_{i_k} = AEF$ in Figure 4.) $\widehat{\Phi}_n$ maps $W \setminus \bigcup_{j=0}^{3} T^n_{k+j}$ into $T^{n-1}_{i_k}$ (i.e., the undistorted

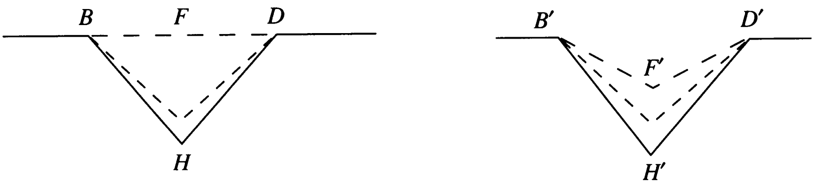

FIGURE 3

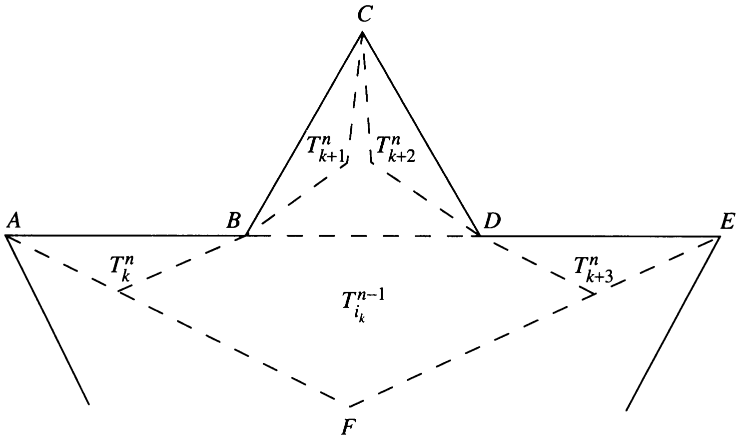

FIGURE 4

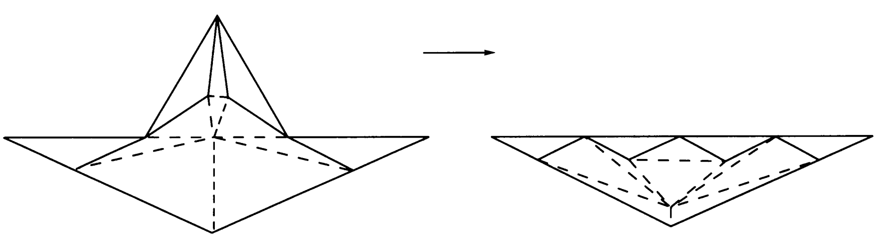

FIGURE 5

boundary triangle of the map $\widehat{\Phi}_{n-1}$; Figure 4) by piecewise linear maps on the triangulation of $W\setminus\bigcup_{j=0}^{3} T^n_{k+j}$ into the triangulation of $T^{n-1}_{i_k}\setminus\bigcup_{j=0}^{3} T^{n'}_{k+j}$ shown in Figure 5.

Since T^n_{k+j} has base 3^{-n} and height $3^{-n}/2\tan\alpha$, and since T^n_{k+j} is mapped to $T^{n'}_{k+j}$ which has base $\frac{3}{4}3^{-n}$ and height $\frac{3}{8}3^{-n}\tan\alpha$, the triangle C' (see Figure 6 on the next page) which is the image of $W-(T^n_{k+1}\cup T^n_{k+2})$ under $\widehat{\Phi}_n$ can be of height $\frac{3}{4}3^{-n}\tan\alpha<\frac{1}{2}3^{-(n-1)}\tan\alpha=$ height of triangle $T^{n-1}_{i_k}$. So there is enough room for the requisite triangulations of the interior regions: $C \to C'$ and

$$T_{i_k}^{n-1}\backslash\{C \cup T_k^n \cup T_{k+3}^n\} \to T_{i_k}^{n-1}\backslash\{C' \cup T_k^{n'} \cup T_{k+3}^{n'}\}$$

(Figure 6).

The parts of $\partial T_{i_k}^{n-1}$ labeled F_1 and F_2 in Figure 6 are mapped to F_1' and F_2'; these sides must be elongated by $\widehat{\Phi}_n$ since $\partial T_{k+j}^n \cap \partial T_{i_k}^{n-1}$ is compressed. This produces a band of second generation distortion by $\widehat{\Phi}_n$ (see Figures 7, 8, and 9).

These bands lie in areas distorted by $\widehat{\Phi}_{n-1}$ or $\widehat{\Phi}_{n-k}$ for $k > 1$. However each band of second generation distortion lies at a distance of approximately $\frac{3}{4}3^{-n}\tan\alpha$ from ∂S^{n-1} and its dimension is less than 3^{-n}. The bands which will be distorted by $\widehat{\Phi}_{n-1}$ lie at a distance approximately $3^{-(n-1)}$ from ∂S^{n-1} so there will be no overlap (Figure 8).

Now set $\widehat{\Phi}_n =$ identity map on the interior of S_n (region inside the double line in Figure 7). $\widehat{\Phi}_n$ will consist of distorting linear maps on regions left undistorted by $\widehat{\Phi}_k$ (boundary triangles) or on regions which will be distorted by only *one* $\widehat{\Phi}_k$ $\forall k < n$. Figure 7 shows double distortion occurring on successive

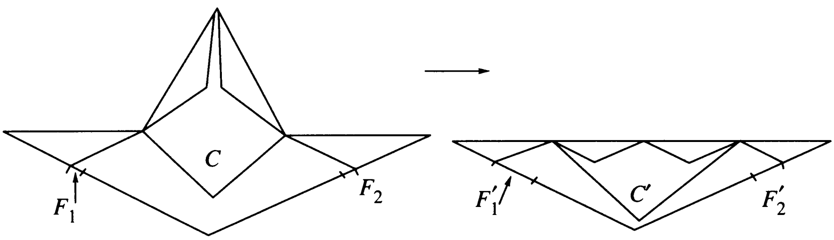

FIGURE 6

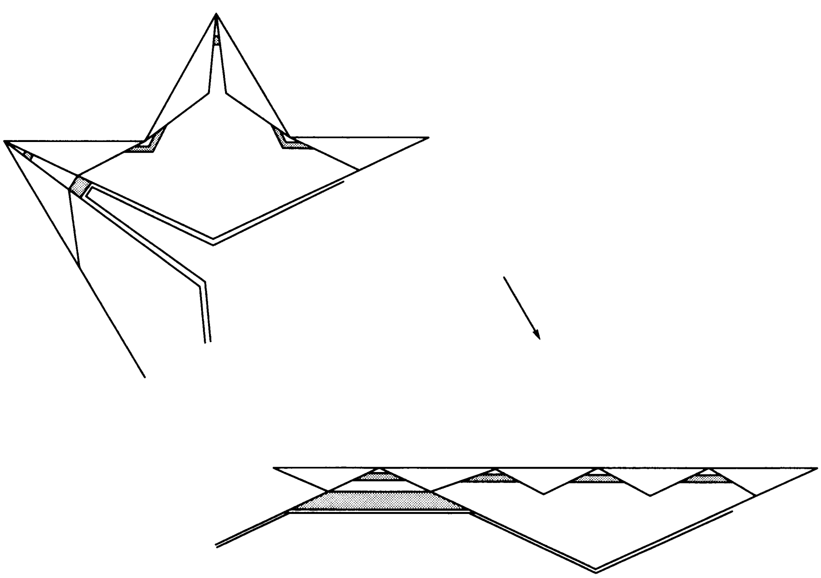

FIGURE 7

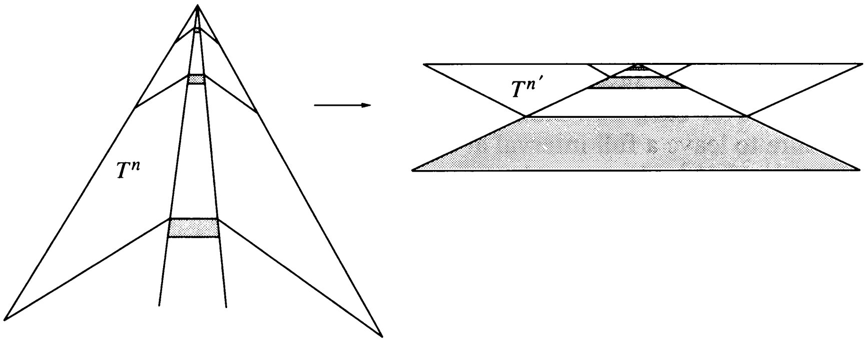

FIGURE 8

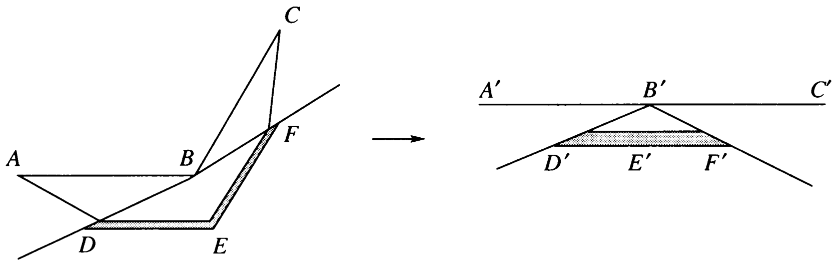

FIGURE 9

generations; the region inside the double line is left unaltered by $\hat{\varphi}_n$ but lies within the triangle of first generation distortion for $\hat{\varphi}_{n-1}$.

The first generation distortion areas are disjoint for each $\widehat{\Phi}_n$ $\forall n$, so there are at most two maps $\widehat{\Phi}_k$ and $\widehat{\Phi}_l$ which will distort over the same region $\forall k, l$.

Now suppose $\Phi_{n-1}: S_{n-1} \to T = S_0 =$ the unit equilateral triangle. Set $\Phi_n = \Phi_{n-1} \circ \widehat{\Phi}_n: S_{n-1} \to T$. $\Psi \circ \Phi_n$ will map S_n to D if Ψ is the canonical conformal map of T to the unit disk D. Then $\Phi = \lim_{n\to\infty} \Psi \circ \Phi_n$ will be a map from S to D such that Φ is piecewise linear on $\operatorname{int} S$ and $\Phi|_{\partial S}$ is essentially the Kaufman-Wu map on ∂S [6]. By construction Φ is quasi-conformal on $\operatorname{int} S$ and $\Phi|_{\partial S}$ creates a measure μ on ∂S such that $\mu(E) = |\Phi(E)| =$ Lebesgue measure of the set $\Phi(E)$. μ has ∂S as its support, i.e., the support of Hausdorff dimension $\log 4/\log 3$.

Results of Caffarelli, Fabes, Mortola, and Salsa [2] and Jerison and Kenig [4] show that any elliptic operator L has a unique measure associated to it on any NTA domain. The operator obtained from pulling the Laplacian Δ on D back to S by the map Φ^{-1} is a strictly elliptic second-order divergence form operator with coefficients bounded and measurable [1]. By the uniqueness of elliptic measure on NTA domains $d\omega_L^x(y) = P(\Phi^{-1}(z), \Phi^{-1}(w))\, d\mu(y)$, where P is the Poisson kernel on D, $x = \Phi^{-1}(z) \in S$, and $y = \Phi^{-1}(w) \in \partial S$. Thus $d\omega_L^x$ has support of Hausdorff dimension $\log 4/\log 3 > 1$.

To show that elliptic measure can have Hausdorff dimension of $2 - \varepsilon$ one must construct a quasi-disk which will allow the same kind of boundary function and piecewise linear map on the interior. The following construction was suggested by Peter Jones. Take a rectangle $k \times 1$ and construct new rectangles to give $k^2 - 1$ new sides of height $\frac{1}{4}$ as shown in Figure 10—construct new sides outward from the interior of the rectangle—this gives an additional length of $\frac{1}{4}(k^2 - 1) + k : k$ ratio on the long sides (see Figure 10). Now repeat the construction over each new side of length $\frac{1}{4}$ and the two ends of length 1—making sure to leave a full interval at each end (see (a) in Figure 10). The same construction of $(k^2 - 1)$ new sides over each long side of each previously added rectangle is repeated to give $(k^2 - 1)/2$ new rectangles of length $= \frac{1}{4}$ width of previous generation rectangles and width; length ratio $1 : k$. The construction is repeated infinitely many times; new rectangles are also made on the short ends of the previous generation rectangles (Figure 13). The limit domain is a quasi-disk.

The ratio of added length to old length is equal to $\frac{1}{4}(k^2 - 1 + 4k)/k$. (This is the ratio on the long side of any new rectangle. The construction over the short ends can also be taken to be $\frac{1}{4}(k^2 + 4k - 1)/k$ by considering a short end to be four long sides of the next generation.) On any such regularly constructed quasi-circle with an added length to old length ratio of M/N at each stage one can construct a boundary function f_n which collapses the boundary of Q_n into the boundary of Q_{n-1} on a ratio of M/N, and the limit map $f = \lim_{n\to\infty} f_n$ will form a measure on the quasi-circle which is supported on the whole boundary. The map f can be used to show that the Hausdorff dimension of the quasi-circle ∂Q equals

$$\lim_{n\to\infty} \frac{\log M^{-n}}{\log N^{-n}} = \frac{\log M}{\log N}.$$

So since $\log \frac{1}{4}(k^2 - 1 + 4k)/\log k = 2 - \varepsilon(k)$, where $\varepsilon(k) \to 0$ as $k \to \infty$, ∂Q can be made to have Hausdorff dimension arbitrarily close to 2 [6].

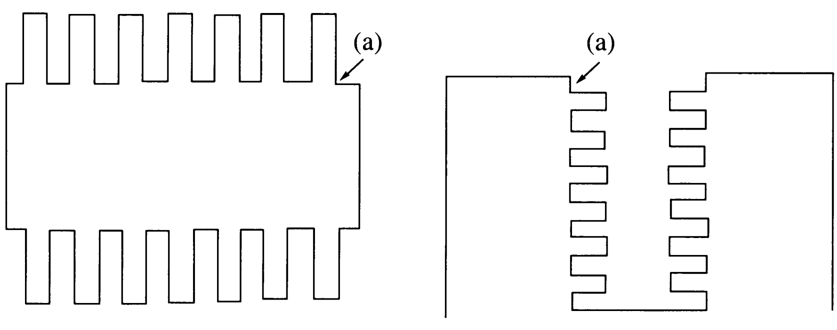

FIGURE 10

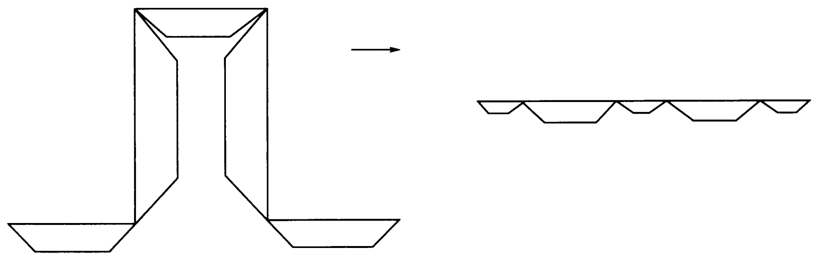

FIGURE 11

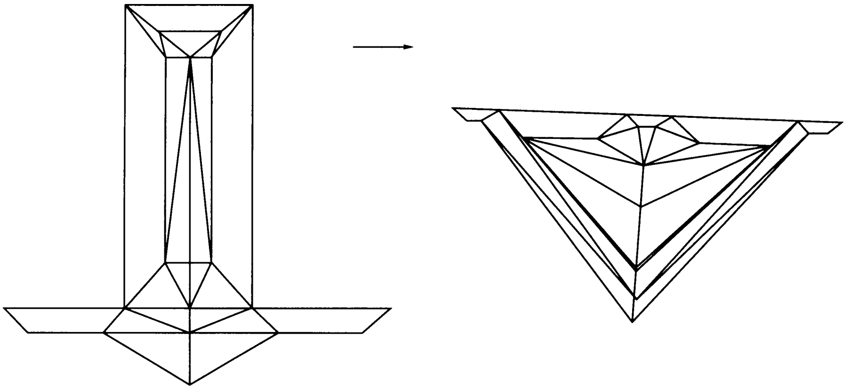

FIGURE 12

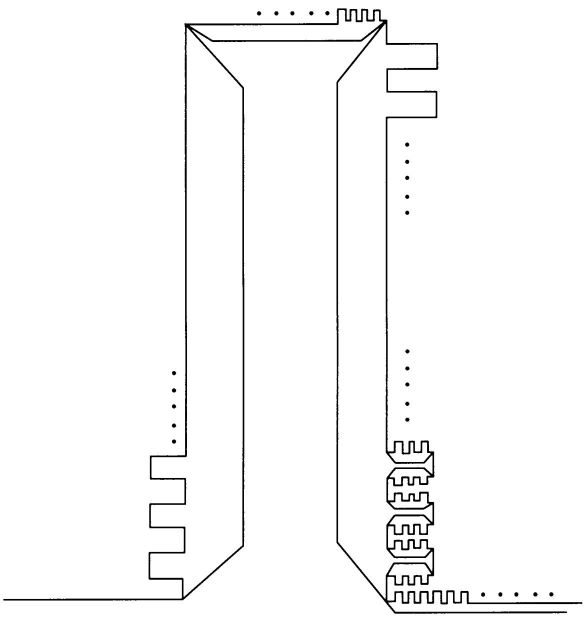

FIGURE 13

Now one can again form piecewise linear maps on Q_n with trace f_n by the same process as above; using triangulation and keeping areas undistorted near the boundary (Figures 11 and 12). The construction is analogous to the construction on the snowflake domain. As before $\widehat{\Phi}_{n+1}$ maps Q_{n+1} into Q_n so that $Q_{n+1}\backslash Q_n$ is mapped inside the boundary quadrilaterals of Q_n (see Figure 13).

On the quadrilaterals near ∂Q_{n+1} $\widehat{\Phi}_{n+1}$ maps by collapsing in a ratio of $\frac{1}{4}(k^2+1+4k)/k$ and as above there will be bands of distortion at the quadrilateral ends which lie at a distance approximately $1/4^{n-2}k^n$ from ∂Q_n. Thus any interior region will be distorted by at most two of the maps $\widehat{\Phi}_n$, $n=1, \infty$. An elementary calculation shows that it is possible to confine first generation distortion to the interior of the boundary quadrilaterals of Q_{n-1}.

The limit map $\Phi = \lim_{n\to\infty} \Phi_n$, where Φ_n maps Q_n to Q_0, composed with the canonical conformal map of the rectangle Q_0 to the unit disk will give a quasi-conformal map of Q to D. The elliptic measure is then obtained exactly as above by pulling the Laplacian back to Q by Φ^{-1}, and looking at the associated L-operator. Its elliptic measure will have support of Hausdorff dimension $2 - \varepsilon$.

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