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PRINCIPAL EIGENVALUES FOR INDEFINITE-WEIGHT ELLIPTIC PROBLEMS IN R^n

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ABSTRACT. We consider the problem $-\Delta u = \lambda g u$ in R^n, $u \to 0$ at ∞ with g a function that changes sign. Under suitable growth conditions on g we show that this problem has an eigenvalue λ with a positive solution u, as well as countably many other eigenvalues.

1. INTRODUCTION

Consider the problem

$$\begin{aligned} -\Delta u &= \lambda g u \quad \text{in } R^n, \\ u &\to 0 \quad \text{as } |x| \to \infty \end{aligned} \tag{1}$$

where Δ denotes the formal Laplace operator and g is a smooth bounded function that changes sign, i.e., g is an indefinite weight function. A principal eigenvalue of (1) is a λ for which (1) has a positive solution. The problem of the existence of such λ was recently considered by Brown, Cosner, and Fleckinger [5]. They showed, in particular, the following results:

Theorem. (a) *Suppose $n \geq 2$ and there exist $R > 0$, $K > 0$ such that $g(x) \leq -K$ for $|x| > R$. Then there exists a positive principal eigenvalue of* (1) *with eigenvector in $L^2(R^n)$.*

(b) *Suppose $n \geq 3$ and there exist $K > 0$, $\alpha > 1$ such that $|g(x)| \leq K(1 + |x|^2)^{-\alpha}$. Then there exists a positive principal eigenvalue of* (1).

In this paper we employ Sobolev's embedding theorems and eigencurve arguments [2, 4] to show the following extensions of the above results:

Theorem 1. *Suppose $n \geq 3$ and $g_+ \in L^{n/2}$. Then there exist infinitely many eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ of* (1) *with λ_1 a principal eigenvalue.*

Corollary 1. *If $n \geq 3$ and $g \in L^{n/2}$ then there exist two principal eigenvalues of* (1) *of opposite sign and infinitely many others $\lambda_j^{\pm} \to \pm\infty$ as $j \to \infty$.*

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Next, following the definition in [3], we set

$$M_{\alpha,1}(\omega) = \sup_{x \in R^n} \int_{|y|<1} |\omega(x-y)|\, |y|^{\alpha-n},$$
$$M_{\alpha,1}(\omega, \Omega) = M_{\alpha,1}(\omega|_\Omega),$$

with $\omega|_\Omega(x) = \omega(x)$ if $x \in \Omega$ or $\omega|_\Omega(x) = 0$ if $x \notin \Omega$. Let $B_R = \{x \mid |x| < R\}$. We will show

Theorem 2. *Suppose* $n \geq 3$, $g = g_1 - g_2$ *with* $g_1 > 0$, $g_2 \geq \varepsilon > 0$, *and* $M_{\alpha,1}(g_1, R^n - B_R) \to 0$ *as* $R \to \infty$ *for some* $0 < \alpha < 2$. *Then the conclusion of Theorem* 1 *holds.*

The procedure employed to obtain these results is very briefly as follows: Let $A\varphi = g_1\varphi$ denote a compact map on the space with norm $(-\Delta\varphi + \lambda g_2\varphi, \varphi)$ for any $\lambda > 0$. This structure ensures the existence of eigenfunctions u and eigenvalues $k(\lambda) : -\Delta u + \lambda g_2 u = k(\lambda) g_1 u$. We next show that k is continuous and that for some $0 < \lambda < \mu$ we have $k(\lambda) - \lambda > 0 > k(\mu) - \mu$, whence $k(\lambda) = \lambda$ for some λ. Observe that we do not require that A be compact on the space associated with $(-\Delta\varphi, \varphi)$, although this may be the case in some specific examples. Consequently, results for $n = 2$ can also be obtained by these arguments. Specifically

Theorem 3. *Suppose* $n = 2$ *and* $g = g_1 - g_2$ *with* $g_1 > 0$, $g_2 \geq \varepsilon > 0$, $g_1 \in L^{3/2}(R^2)$. *Then the conclusion of Theorem* 1 *holds.*

We observe that Corollary 1 is identical in form to the classical result for bounded domains (see, e.g., [2]), but Theorems 2 and 3 are not. Indeed, under the conditions of Theorem 2 or 3, problem (1) cannot have a principal eigenvalue $\lambda \leq 0$ since, in this case, $-\Delta u - \lambda g u = 0$ is oscillatory and there are no positive solutions of (1) near $|x| = \infty$.

There clearly is a connection between these results and the criticality of $-\Delta u - \lambda_1 g u$. We comment on this briefly at the end of the paper.

In conclusion, we observe that applications and further references on indefinite weight problems may be found in [2, 5]. We have considered $-\Delta$ in R^n for convenience. Extension of the results to the more general case of a uniformly elliptic operator in unbounded domains is immediate.

2. Results

We consider first the case $n \geq 3$. Without loss of generality, set $g = g_1 - g_2$ with $g_1, g_2 > 0$ and $g_1, g_2 \in L^\infty$. Let $\lambda > 0$ be chosen arbitrarily and define E_λ to be the completion of $C_0^\infty(R^n)$ with respect to the norm

$$\|u\|_\lambda^2 = \int_{R^n} |\nabla u|^2 + \max(\lambda g_2, \omega) u^2$$

where $\omega = K/(1+|x|^2)$ with K small and positive. As is well known, Hardy's Inequality implies $\| \ \|_\lambda \sim \| \ \|_{l_\lambda}$ where

$$\| \ \|_{l_\lambda}^2 = \int_\Omega |\nabla u|^2 + \lambda g_2 u^2.$$

It is apparent that $\| \ \|_{\lambda_1} \sim \| \ \|_{\lambda_2}$ for any positive λ_1, λ_2, so we denote E_λ, $\| \ \|_\lambda$ merely by E, $\| \ \|$ in the following. Finally, we let L_ω^2 represent the weighted

L^2 space with weight ω and represent by $(\ ,\ )$, $(\ ,\ )_{l_\lambda}$, $(\ ,\ )_0$ the inner products associated with $\|\ \|$, $\|\ \|_{l_\lambda}$, $L^2(R^n)$, respectively.

We recall the following preliminary results

Lemma 0. (1) *If $u_n \to u$ in E then $u_n \to u$ in $H^1(B_R)$ for any ball $B_R \subset R^n$.*

(2) $E \hookrightarrow L^{2n/(n-2)}$, $E \hookrightarrow L^2_\omega$.

(3) *Let $g_1 \in L^{n/2}$. The operator $A\colon E \to E^* \hookrightarrow E$ given by $A(\varphi) = g_1\varphi$, i.e., $(g_1\varphi, \psi)_0 = (A\varphi, \psi)_{l_\lambda}$ for all $\psi \in E$, is compact and selfadjoint.*

(4) *If $u \in E$ satisfies $(u, \psi)_{l_\lambda} = k(g_1 u, \psi)_0$ for some constant k and any $\psi \in C_0^\infty(R^n)$, then $u \in C^2$, $-\Delta u + \lambda g_2 u = k g_1 u$, and $u \to 0$ at ∞.*

Proof. The proof of (1) is immediate since $\|\ \| \sim \|\ \|_{l_\lambda}$ and, in any ball B_R, $\max(\lambda g_2, \omega) \geq \omega > c > 0$ for some constant $c = c(R)$. Similarily for (2), we first employ Sobolev's Inequality [6, Theorem 7.10, equation (7.26)] and obtain for $\varphi \in C_0^\infty(R^n)$

$$\|\varphi\|_{2n/(n-2)} \leq C_1\|\nabla\varphi\|_2 \leq C_1\|\varphi\|.$$

Since C_1 only depends on n, the same estimate holds for $u \in E$ and we have $E \hookrightarrow L^{2n/(n-2)}$. The second embedding follows in the same way from Hardy's Inequality, specifically,

$$\int_{R^n} \omega\varphi^2 \leq C_2 \int_{R^n} |\nabla\varphi|^2$$

with C_2 also only a function of n and $\varphi \in C_0^\infty(R^n)$. Next for (3), observe that the Holder Inequality

$$\int_{R^n} g\phi\psi \leq \|g\|_{n/2}\ \|\phi\|_{2n/(n-2)}\ \|\psi\|_{2n/(n-2)}$$

is valid for $g \in L^{n/2}(R^n)$, ϕ, $\psi \in E$, and thus A is defined. Suppose first that g_1 has compact support. (1) and (2) imply that $A(\varphi)$ is compact since H^1 is compactly embedded in $L^p (p < 2n/(n-2))$ for any ball. In general, set $g_r = g_1$ if $|x| < r$ and $g_r = 0$ otherwise and let $A_r\varphi \triangleq g_r\varphi$. Then $\|A_r\varphi - A\varphi\| \leq K\|g_r - g_1\|_{n/2}\|\varphi\|$, i.e., $A_r \to A$ in the operator norm and A is also compact. Note that, since A is obviously symmetric and defined on the whole of E, it is selfadjoint. Finally for (4), if $u \in E$ then $u \in H^1_{\mathrm{loc}}$ and thus u is a weak solution of $-\Delta u + \lambda g_2 u = k g_1 u$. We apply [6, p. 243] and conclude that $u \in C^2$ since g_1, g_2 are smooth. In conclusion we note that by (2), $u \in L^{2n/(n-2)}$, and we recall [6, p. 194]

$$|u(x)| \leq K[\|u\|_{L^p(B_2(x))}]$$

for some constant K and that $p = 2n/(n-2)$, whence $u \to 0$ at ∞.

Proof of Theorem 1. We show in detail only the existence of λ_1. The existence of the other λ_i follows with obvious changes. We apply the Spectral Theorem and conclude for $\lambda > 0$ that

$$\sup_{\varphi \in E} \frac{(g_1\varphi, \varphi)_0}{\|\varphi\|^2_{l_\lambda}} \triangleq [k(\lambda)]^{-1}$$

is achieved by an eigenfunction $u \in E$ corresponding to the largest eigenvalue of A. Since $g_1 > 0$, we also have

$$k(\lambda) = \inf_{\varphi \in E}\left[\frac{\|\varphi\|^2_{l_\lambda}}{(g_1\varphi, \varphi)_0}\right] = \inf_{\varphi \in C_0^\infty}\left[\frac{\|\varphi\|^2_{l_\lambda}}{(g_1\varphi, \varphi)_0}\right].$$

By Lemma 0(4), we conclude that u satisfies $L_\lambda u = k(\lambda) g_1 u$, $u \to 0$ at ∞, where L_λ denotes the formal differential expression associated with l_λ. Since the functional being minimized is quadratic, and observing that if u belongs to E then so do u^+, u^-, we may assume $u \geq 0$, for otherwise we replace u by $|u|$, whence $u > 0$ by the maximum principle since zero cannot be a minimum of a nonnegative nontrivial solution of $L_\lambda u = k(\lambda) g_1 u$. Observe that $(g_1\varphi, \varphi)_0 \leq C\|g_1\|_{n/2}\|\nabla\varphi\|_0^2 \leq C\|g_1\|_{n/2}\|\varphi\|_{l_\lambda}^2$, where C is the Sobolev Embedding constant. From this, we obtain that $k(\lambda) > \lambda$ for λ small. Next we observe that since

$$k(\lambda) - \lambda = \inf_{\varphi \in C_0^\infty} \frac{(-\Delta\varphi + \lambda g\varphi, \varphi)_0}{(g_1\varphi, \varphi)_0}$$

and $g < 0$ in some subset, $k(\lambda) < \lambda$ for λ large. We also note that $k(\lambda)$ is locally Lipschitz in R^+. To see this, let $h > 0$ and observe

$$k(\lambda + h) = \inf_{\varphi \in C_0^\infty} \frac{(-\Delta\varphi + (\lambda + h) g_2\varphi, \varphi)_0}{(g_1\varphi, \varphi)_0}$$

$$\leq \left(\frac{\lambda + h}{\lambda}\right) \inf_{\varphi \in C_0^\infty} \frac{(-\Delta\varphi + \lambda g_2\varphi, \varphi)_0}{(g_1\varphi, \varphi)_0} = \left(1 + \frac{h}{\lambda}\right) k(\lambda).$$

Since $k(\lambda + h) \geq k(\lambda)$ by definition, we obtain

$$|k(\lambda_1) - k(\lambda_2)| \leq k(\lambda_1)|\lambda_1 - \lambda_2|/\lambda_1.$$

We conclude that $k(\lambda) = \lambda$ for some $\lambda > 0$ and the result follows.

Proof of Theorem 2. In view of the results given in [3], most of the arguments in the proofs of Lemma 0 and Theorem 1 need only to have $\|g_1\|_{L^{n/2}(B_R)}$, $\|g_1\|_{L^{n/2}(R^n - B_R)}$ substituted by $M_{\alpha,1}(g_1, B_R)$, $M_{\alpha,1}(g_1, R^n - B_R)$, respectively. Note that $E \sim H^1$ here and that for $\lambda > 0$ compactness of g_1 and continuity of $k(\lambda)$ follow as before, as does $k(\lambda) - \lambda \to -\infty$ as $\lambda \to \infty$. To see that $k(\lambda) > \lambda$ for some $\lambda > 0$, observe that the embedding theorems given in [3, Theorem 2.3] show for $\varphi \in C_0^\infty(R^n)$

$$\begin{aligned}
(g_1\varphi, \varphi)_0 &= \int_{B_R} g_1\varphi^2 + \int_{R^n - B_R} g_1\varphi^2 \\
&\leq C[\|\nabla\varphi\|_{L^2(B_R)}^2 + \|\varphi\|_{L^2(B_R)}^2] \\
&\quad + CM_{\alpha,1}(g_1, R^n - B_R)^2[\|\nabla\varphi\|_{L^2(R^n)}^2 + \|\varphi\|_{L^2(R^n)}^2] \\
&\leq C\left[(1 + R^2) + \frac{M_{\alpha,1}(g_1, R^n - B_R)^2}{\lambda}\right]\|\varphi\|_{l_\lambda}^2
\end{aligned}$$

for some constant C independent of λ and $0 < \lambda$ small. Here we have employed the equivalence $\| \ \|_\lambda \sim \| \ \|_{l_\lambda}$. We thus conclude:

$$k(\lambda) \geq \left[C(1 + R^2) + \frac{CM_{\alpha,1}(g_1, R^n - B_R)^2}{\lambda}\right]^{-1}.$$

Finally, we select R so that $CM_{\alpha,1}(g_1, R^n - B_R)^2 < 1$ and, consequently, obtain $k(\lambda) > \lambda$ for some $0 < \lambda$ small enough. To see the decay of the eigenfunction observe that $u \in L^2$ and $L_\lambda u = -\Delta u - \lambda g u = 0$. Since the

essential spectrum of the selfadjoint operator $L_\lambda\colon L^2 \to L^2$ thus generated is contained in the interval $[\lambda\varepsilon, \infty)$, we conclude that u is an eigenvector of L_λ with eigenvalue below the essential spectrum and the decay follows.

Proof of Theorem 3. Let $\varphi \in C_0^\infty(R^2)$, $\psi \in C_0^\infty(-a, a)$ with $\int_{-a}^a \psi^2 = 1$, and $a > 0$ to be chosen below. Set

$$\tau(x, y, z) = \varphi(x, y)\psi(z), \qquad h(x, y, z) = g_1(x, y) \cdot \chi_{(-a,a)}(z)$$

where $\chi_{(-a,a)}(z) = 1$ if $|z| < a$ and 0 otherwise, and note $\tau \in C_0^\infty(R^2 \times (-a, a))$. We apply the embedding estimates for R^3 and conclude

$$(h\tau, \tau)_0 \le C\|h\|_{L^{3/2}(R^3)}\|\nabla\tau\|^2_{L^2(R^3)}. \tag{2}$$

Expanding (2), we obtain

$$\begin{aligned}&\Big(\int_{R^2} g_1\varphi^2\Big)\Big(\int_{-a}^a \psi^2\Big)\\ &\quad\le C\Big(\int_{R^2} g_1^{3/2}\Big)^{2/3}(2a)^{2/3}\left[\Big(\int_{R^2}|\nabla\varphi|^2\Big)\Big(\int_{-a}^a\psi^2\Big) + \Big(\int_{R^2}\varphi^2\Big)\Big(\int_{-a}^a(\psi')^2\Big)\right],\end{aligned}$$

i.e.,

$$\int_{R^2} g_1\varphi^2 \le C\Big(\int_{R^2} g_1^{3/2}\Big)^{2/3} a^{2/3}\left[\int_{R^2}|\nabla\varphi|^2 + \Big(\int_{R^2}\varphi^2\Big)\Big(\int_{-a}^a(\psi')^2\Big)\right]. \tag{3}$$

This inequality clearly also applies to any smooth normalized (in $L^2(-a, a)$) function ψ with $\psi(-a) = \psi(a) = 0$. In particular, we may choose ψ to be the first eigenfunction of $-v'' = \theta v$, $v(-a) = v(a) = 0$, i.e., $\psi = \sin[\pi(x+a)/2a]/(\sqrt{a})$. Substituting this ψ into equation (3) yields in R^2

$$(g_1\varphi, \varphi)_0 \le C\|g_1\|_{L^{3/2}(R^2)} \cdot a^{2/3}\left[\int_{R^2}|\nabla\varphi|^2 + \frac{\pi^2}{4a^2}\varphi^2\right].$$

We now choose $\lambda\varepsilon = \pi^2/4a^2$ and obtain

$$(g_1\varphi, \varphi)_0 \le C\|g_1\|_{L^{3/2}(R^2)}\lambda^{-1/3}[\|\varphi\|^2_{I_\lambda}].$$

Compactness arguments, continuity of $k(\lambda)$ and $k(\lambda) - \lambda \to -\infty$, and decay of any eigenfunction follow as before. Finally we observe that $k(\lambda) \ge [C\|g_1\|_{L^{3/2}(R^2)}]^{-1}\lambda^{1/3}$, whence $k(\lambda) > \lambda$ for $0 < \lambda$ small, and, therefore, $k(\lambda) = \lambda$ for some $0 < \lambda$.

We conclude by briefly and heuristically connecting our results to the question of the criticality of the operator formally given by $L\varphi = -\Delta\varphi - \lambda_1 g\varphi$, see [1, 7, 8] for definitions and background discussion of this and related concepts. Specifically, we point out that above we have constructed critical operators L with decaying positive solutions. We recall that critical or subcritical operators need not have such solutions, e.g., $-\Delta$ in R^n, so that no conclusions can be drawn in general about the decay of positive solutions of critical or subcritical operators. To see the criticality of L, observe that $(L\varphi, \varphi)_0 \ge 0$ for any $\varphi \in C_0^\infty$, whence L is either critical or subcritical. Suppose that L satisfies the

"λ-property": for some function $h > 0$ and any $\varphi \in C_0^\infty$ we have $(L\varphi, \varphi)_0 \geq (h\varphi, \varphi)_0$. In such a case, we construct a positive function v such that $Lv - hv = 0$ and apply Picone's Identity to get

$$\int_{R^n} v^2 \left|\nabla\left(\frac{\varphi}{v}\right)\right|^2 = \int_{R^n} \varphi L\varphi - \int_{R^n} \frac{\varphi^2}{v} Lv,$$

i.e.,

$$\int_B \varphi^2 h \leq \int_{R^n} \varphi^2 h + \int_{R^n} v^2 \left|\nabla\left(\frac{\varphi}{v}\right)\right|^2 = \int_{R^n} \varphi L\varphi$$

for any fixed ball B and $\varphi \in C_0^\infty$.

We let $\varphi \to u$ in E and conclude

$$\int_B u^2 h \leq \int_{R^n} uLu = 0,$$

i.e., $u \equiv 0$. This shows that L does not satisfy the "λ-property" and thus must be critical.

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