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A REMARK ON SAKAI'S QUADRATIC RADON-NIKODYM THEOREM

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ABSTRACT. Sakai's Radon-Nikodym theorem (in a quadratic form) for normal states on a von Neumann algebra is considered. We show that the conclusion of this theorem follows from a much weaker order assumption on involved states.

1. INTRODUCTION

Let ψ be a faithful normal state on a von Neumann algebra M. Sakai's Radon-Nikodym theorem (in a quadratic form) [12] states that if $\varphi \in M_*^+$ satisfies $\varphi \leq l\psi$ for some $l > 0$ then there exists a (unique) positive operator h in M $(0 \leq h \leq l^{1/2}1)$ such that $\varphi(x) = \psi(hxh)$, $x \in M$. We will point out that the same conclusion follows from a much weaker assumption.

In [8, 11] a necessary and sufficient condition for φ to admit a (bounded) quadratic Radon-Nikodym derivative was found. However, in practical applications checking this condition seems difficult. On the other hand, an unbounded quadratic Radon-Nikodym derivative was studied in [13]. So far the following practical and basic question has been untouched: Does the existence of a (bounded) quadratic Radon-Nikodym derivative follow from the assumption on the order determined by the natural cone $\mathscr{P}^\natural$ [1, 2, 6]? Based on the result [5] we will show that the answer is affirmative (even under a much weaker assumption).

2. MAIN RESULT

Let $L^p(M)$ be the Haagerup L^p-space [7], and assume that φ, $\psi \in M_*^+$ correspond to h_φ, $h_\psi \in L^1(M)_+$, respectively. The usual assumption $\varphi \leq l\psi$ in Sakai's theorem of course means $h_\varphi \leq lh_\psi$ (as τ-measurable operator—here, τ is the canonical trace on the crossed product $M \rtimes_{\sigma^\psi} \mathbb{R}$). Let us assume the following weaker condition [3]: for some $\varepsilon > 0$ the Connes Radon-Nikodym cocycle $f(t) = (D\varphi : D\psi)_t$ $(t \in \mathbb{R})$ extends to a bounded $(\sup_z \|f(z)\| \leq l)$, σ-ω continuous function on the strip $-\varepsilon/2 \leq \operatorname{Im} z \leq 0$ that is analytic in the interior.

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For a vector $\xi \in D = \mathscr{D}(h_\varphi^{\varepsilon/2}) \cap \mathscr{D}(h_\psi^{\varepsilon/2})$, we consider the two functions

$$g(z) = h_\varphi^{iz}\xi, \qquad h(z) = f(z)h_\psi^{iz}\xi.$$

Each of them is a bounded continuous function on the strip $-\varepsilon/2 \le \operatorname{Im} z \le 0$ that is analytic in the interior. Since $f(t) = (D\varphi : D\psi)_t = h_\varphi^{it}h_\psi^{-it}$, $t \in \mathbb{R}$, we have $g(z) = h(z)$ for $z = t \in \mathbb{R}$. Uniqueness of analytic continuation shows $g(-i\varepsilon/2) = h(-i\varepsilon/2)$, that is,

$$h_\varphi^{\varepsilon/2}\xi = uh_\psi^{\varepsilon/2}\xi, \qquad \xi \in D,$$

with $u = f(-i\varepsilon/2) \in M$, $\|u\| \le l$. Since D is a common core for the (τ-measurable) operators $h_\varphi^{\varepsilon/2}$ and $uh_\psi^{\varepsilon/2}$, we conclude that

$$h_\varphi^{\varepsilon/2} = uh_\psi^{\varepsilon/2}, \qquad h_\varphi^{\varepsilon} = h_\psi^{\varepsilon/2}u^*uh_\psi^{\varepsilon/2} \le \|u\|^2 h_\psi^\varepsilon \le l^2 h_\psi^\varepsilon.$$

Furuta's inequality [5] states that, whenever bounded operators A, B satisfy $A \ge B \ge 0$, we get

$$A^{(p+2r)/q} \ge (A^rB^pA^r)^{1/q} \tag{1}$$

for $r \ge 0$, $p \ge 0$, $q \ge 1$, $(1+2r)q \ge p+2r$. Since this inequality remains valid for τ-measurable operators (see the next section), with $p = 1/\varepsilon$, $r = p/2$, $q = 2$ we get

$$(h_\psi^{1/2}h_\varphi h_\psi^{1/2})^{1/2} \le l^{1/\varepsilon}h_\psi. \tag{2}$$

Notice that

$$\begin{cases} (h_\psi^{1/2}h_\varphi h_\psi^{1/2})^{1/2} = |h_\varphi^{1/2}h_\psi^{1/2}|, \\ \operatorname{tr}(h_\varphi^{1/2}h_\psi^{1/2}x) = \operatorname{tr}(xh_\varphi^{1/2}h_\psi^{1/2}) = (xh_\varphi^{1/2}|h_\psi^{1/2})_{L^2(M)}, \qquad x \in M. \end{cases} \tag{3}$$

Let $(M, \mathscr{H}, J, \mathscr{P}^\natural)$ be a standard form of M [1, 2, 6], and ξ_φ, ξ_ψ be the unique implementing vectors in $\mathscr{P}^\natural$ for φ, ψ, respectively.

Lemma 1 [8, Theorem A]. *There exists a (unique) positive operator h in M such that $\varphi(x) = \psi(hxh)$, $x \in M$, if and only if the absolute value part $|\chi_\varphi|$ of the polar decomposition of $\chi_\varphi = (\cdot\xi_\varphi|\xi_\psi) \in M_*$ satisfies $|\chi_\varphi| \le l\psi$ for some $l > 0$. Furthermore, in this case h is exactly $|(D|\chi_\varphi| : D\psi)_{-i/2}|^2$ (so that $0 \le h \le l1$).*

In L^p-space languages the vectors ξ_φ, $\xi_\psi \in \mathscr{P}^\natural$ are $h_\varphi^{1/2}$, $h_\psi^{1/2} \in L^2(M)_+$, respectively. Hence (3) shows that $\chi_\varphi \in M_*$, $|\chi_\varphi| \in M_*^+$ correspond to $h_\varphi^{1/2}h_\psi^{1/2} \in L^1(M)$, $(h_\psi^{1/2}h_\varphi h_\psi^{1/2})^{1/2} \in L^1(M)_+$, respectively. Therefore, (2) means $|\chi_\varphi| \le l^{1/\varepsilon}\psi$, and Lemma 1 shows the main result of the article.

Theorem 2. *Let ψ be a faithful normal state on a von Neumann algebra M. Assume that $\varphi \in M_*^+$ satisfies: for some $\varepsilon > 0$, $f(t) = (D\varphi : D\psi)_t$ $(t \in \mathbb{R})$ extends to a bounded ($\sup_z \|f(z) \le l$), σ-ω continuous function on the strip $-\varepsilon/2 \le \operatorname{Im} z \le 0$ that is analytic in the interior. Then there exists a unique positive operator h in M $(0 \le h \le l^{1/\varepsilon}1)$ such that $\varphi(x) = \psi(hxh)$, $x \in M$.*

For $\varepsilon = 1$ the theorem is exactly the usual version of Sakai's theorem. When $\varepsilon = 1/2$, the assumption is equivalent to $l^2\xi_\psi - \xi_\varphi \in \mathscr{P}^\natural$ (as was shown in [2]). We thus have shown

Corollary 3. *Assume that the unique implementing vectors ξ_φ, ξ_ψ in the natural cone $\mathcal{P}^\natural$ satisfy $l\xi_\psi - \xi_\varphi \in \mathcal{P}^\natural$. Then there exists a unique positive operator h in M $(0 \le h \le l1)$ such that $\varphi(x) = \psi(hxh)$, $x \in M$.*

The author does not know (and doubts) if the assumption $l\xi_\psi - \xi_\varphi \in \mathcal{P}^\natural$ guarantees the existence of a bounded Radon-Nikodym derivative in a Jordan form, i.e., $k \in M_+$ satisfying $\varphi(x) = \psi(kx + xk)$, $x \in M$ (see [8, Proposition 3.2.6; 9, Theorem 1.9]). If the answer is affirmative, then we would obtain a different proof of Corollary 3 (because of [8, Proposition 3.2.7]). On the other hand, starting from the same assumption, Araki [1, Corollary, p. 334] showed the following "vector version": there exists a positive operator $k \in M$ satisfying $\xi_\varphi = k\xi_\psi + Jk\xi_\psi$. Based on Araki's result (and without using Furuta's inequality) one can prove Corollary 3 (by making use of techniques in [8, 10, 11]). However, the proof presented in the article (valid under the much weaker assumption $h_\varphi^\varepsilon \le l^2 h_\psi^\varepsilon$) seems easier and more natural.

3. Furuta's inequality for τ-measurable operators

Here we show that (1) remains valid for τ-measurable operators $A \ge B \ge 0$. Using the spectral projections $\{e_\lambda\}$ of A, we set

$$A_n = e_n A (= e_n A e_n) \ge B_n = e_n B e_n \qquad (n = 1, 2, \dots).$$

Since $A_n \ge B_n$ are bounded, (1) implies

$$A^{(p+2r)/q} \ge A_n^{(p+2r)/q} \ge (A_n^r B_n^p A_n^r)^{1/q}. \tag{4}$$

Choose and fix $t > 0$. Let $\mu_t(\,\cdot\,)$ be the "tth" singular number (see [4] for details). We estimate

$$\begin{aligned}
\mu_t(B - B_n) &= \mu_t(B(1-e_n) + (1-e_n)Be_n) \\
&\le \mu_{t/2}(B(1-e_n)) + \mu_{t/2}((1-e_n)Be_n) \\
&\le 2\mu_{t/2}(B(1-e_n)) \\
&\le 2\mu_{t/4}(B^{1/2})\mu_{t/4}(B^{1/2}(1-e_n)) \\
&\qquad\qquad (\text{note } \mu_{t/4}(B^{1/2}) < +\infty \text{ since } B \text{ is } \tau\text{-measurable}) \\
&= 2\mu_{t/4}(B^{1/2})\mu_{t/4}(|B^{1/2}(1-e_n)|) \\
&= 2\mu_{t/4}(B^{1/2})\mu_{t/4}((1-e_n)B(1-e_n))^{1/2} \\
&\le 2\mu_{t/4}(B^{1/2})\mu_{t/4}((1-e_n)A(1-e_n))^{1/2} \quad (\text{since } 0 \le B \le A) \\
&= 2\mu_{t/4}(B^{1/2})\mu_{t/4}(A - A_n)^{1/2}.
\end{aligned}$$

When $n \to +\infty$, $A_n \to A$ in measure, hence, $\mu_{t/4}(A - A_n) \to 0$. From the above estimate, when $n \to +\infty$, $\mu_t(B - B_n) \to 0$ (for each $t > 0$). We thus know $B_n \to B$ in measure. Thanks to Tikhonov's result [14] we conclude that $(A_n^r B_n^p A_n^r)^{1/q} \to (A^r B^p A^r)^{1/q}$ in measure. Therefore, by letting $n \to +\infty$ in (4), we get (1) for τ-measurable operators.

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