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AN ALGEBRAIC PROOF FOR THE SYMPLECTIC STRUCTURE OF MODULI SPACE

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ABSTRACT. Goldman has constructed a symplectic form on the moduli space $\mathrm{Hom}(\pi, G)/G$, of flat G-bundles over a Riemann surface S whose fundamental group is π. The construction is in terms of the group cohomology of π. The proof that the form is closed, though, uses de Rham cohomology of the surface S, with local coefficients. This symplectic form is shown here to be the restriction of a tensor, that is defined on the infinite product space G^π. This point of view leads to a direct proof of the closedness of the form, within the language of group cohomology. The result applies to all finitely generated groups π whose cohomology satisfies certain conditions. Among these are the fundamental groups of compact Kähler manifolds.

INTRODUCTION

This paper deals with the symplectic structure on the space of "representations" $\mathrm{Hom}(\pi, G)/G$. π is a finitely generated discrete group, G is a Lie group, and G acts on the set $\mathrm{Hom}(\pi, G)$ by pointwise conjugation: $(af)(x) = af(x)a^{-1}$. We will make further assumptions on G and π later.

Goldman constructs, in [G], a symplectic form on $\mathrm{Hom}(\pi, G)/G$, when π is the fundamental group of a compact orientable surface of genus $p \geq 1$. The Lie algebra, g, is then required to have a nondegenerate, symmetric, bilinear form, which is invariant under the adjoint action of the group G:

$$K: g \times g \to \mathbb{R}. \tag{1}$$

The symplectic form is defined in terms of the group cohomology of π. Goldman observed that the tangent space to $\mathrm{Hom}(\pi, G)$ at the point $\phi \in \mathrm{Hom}(\pi, G)$ can be identified with the space of cocycles $Z^1(\pi, g_{\mathrm{Ad}\,\phi}) = \{u: \pi \to g \mid u(xy) = u(x) + \mathrm{Ad}\,\phi(x)u(y)$ for every $x, y \in \pi\}$. The tangent to the orbit O_ϕ of ϕ is the space of coboundaries $B^1(\pi, g_{\mathrm{Ad}\,\phi})$. These observations are shown in [G] by looking at tangents to curves in the spaces $\mathrm{Hom}(\pi, G)$ and O_ϕ. A symplectic structure on the quotient $\mathrm{Hom}(\pi, G)/G$ is, by definition, a closed G-invariant 2-form on $\mathrm{Hom}(\pi, G)$, whose degeneracy is precisely in the directions of the G-action. To construct such a form, one takes the cup product on

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$Z^1(\pi, g_{\mathrm{Ad}\,\phi})$, with the pairing (1) acting on the coefficients. This gives a map

$$Z^1(\pi, g_{\mathrm{Ad}\,\phi}) \times Z^1(\pi, g_{\mathrm{Ad}\,\phi}) \to Z^2(\pi, \mathbb{R}). \tag{2}$$

This map is then composed with the quotient $Z^2(\pi, \mathbb{R}) \to H^2(\pi, \mathbb{R})$ and with an isomorphism $H^2(\pi, \mathbb{R}) \xrightarrow{\cong} \mathbb{R}$, to obtain a symplectic form on $\mathrm{Hom}(\pi, G)/G$.

The difficult part of the construction is in showing that the 2-form is closed. For this, Goldman turns to the language of de Rham cohomology, and uses a method of Atiyah and Bott [AB]: Let P be a principal G-bundle over the surface S. The gauge group $\mathscr{G}(P)$ acts on the space of connections $\mathscr{A}(P)$, preserving the subspace $\mathscr{F}(P)$ of flat connections. The space $\mathscr{A}(P)$ of connections on P admits a symplectic structure which reduces to a symplectic structure on $\mathscr{F}(P)/\mathscr{G}(P)$. This is equivalent to Goldman's symplectic structure under an isomorphism

$$\begin{aligned} &\{\text{a connected component of } \mathrm{Hom}(\pi, G)/G\} \\ &\qquad \cong \{\text{a connected component of } \mathscr{F}(P)/\mathscr{G}(P)\}. \end{aligned} \tag{3}$$

(Each connected component of $\mathrm{Hom}(\pi, G)/G$ determines a bundle P over S.)

This paper contains a direct proof for the closedness of the 2-form on $\mathrm{Hom}(\pi, G)$ in the language of group cohomology, without going through the isomorphism (3).

It turns out to be possible to talk about "differential structure" on the infinite product G^π. The Zariski tangents to the space $\mathrm{Hom}(\pi, G)$ are then defined using the natural embedding

$$\mathrm{Hom}(\pi, G) \subset G^\pi. \tag{4}$$

The closedness of the 2-form on $\mathrm{Hom}(\pi, G)$ can then be proved through a computation in G^π. The embedding (4) avoids the choice of generators in π. It therefore leads to much simpler calculations than those that arise from an embedding $\mathrm{Hom}(\pi, G) \subset G^m$, that depends on the choice of m generators in π. Finally, the assumption on π is stated in terms of the "twisted cohomology groups" $H^1(\pi, g_{\mathrm{Ad}\,\phi})$, and it applies to other than just the fundamental groups of surfaces.

The structure of the paper is as follows. Section 1 defines standard concepts connected with differential structure in the category of "real valued ringed spaces." These concepts are well known. Subsection 1.4 contains an example that motivates the indirect definition of a symplectic structure on a quotient space. Section 2 describes the differential structure on the infinite product G^π and on the subspace $\mathrm{Hom}(\pi, G)$. Section 3 contains Goldman's construction of the symplectic form, obtained as the restriction of a tensor on the space G^π, and a proof that this form is closed. Theorem 4 summarizes the properties of the construction.

In §4 we apply this theorem to show that Goldman's construction gives rise to a symplectic structure on the space $\mathrm{Hom}(\pi, G)/G$, where π is the fundamental group of any compact Kähler manifold—not necessarily a surface. This example was suggested by Livne.

1. Differential structure on ringed spaces

In order to define tensors on spaces which are not differential manifolds, we will work in the category of real valued ringed spaces. Many concepts involving differential manifolds can be introduced into this category, consistently with their standard meanings.

1.1. The tangent to a ringed space.

Definition 1.1. A *real valued ringed space* (X, R) is a topological space X with a sheaf R of continuous real valued functions on X. The functions in R will be called *admissible functions.*

This means that to each open set $U \subseteq X$ we attach a ring $R(U)$ of functions on U, containing the constants, and such that if $f: U \to \mathbb{R}$ satisfies $f_{|U_\alpha} \in R(U_\alpha)$ for a covering $\{U_\alpha\}$ of U, then $f \in R(U)$.

Definition 1.2. A *morphism* between the ringed spaces (X_1, R_1) and (X_2, R_2), is a continuous map $F: X_1 \to X_2$ such that for any open subset $U_2 \subseteq X_2$ and for any admissible function $f: U_2 \to \mathbb{R}$, the pullback $f \circ F: U_1 \to \mathbb{R}$ is admissible on $U_1 = F^{-1}U_2$. F is also called a *structure preserving map.*

Definition 1.3. Let (X, R) be a ringed space, and let $p \in X$. The (*Zariski*) *tangent space* T_pX of X at p (relative to the sheaf R) is the vector space $(\mathscr{M}_p/\mathscr{M}_p^2)^*$, where $\mathscr{M}_p$ denotes the space of germs of admissible functions that are defined near p and that vanish at p.

It follows directly from the definitions, that every morphism $F: (X_1, R_1) \to (X_2, R_2)$ induces a linear map $d_pF: T_pX_1 \to T_{F(p)}X_2$, in a functorial manner.

Example 1.1. Taking the sheaf $C^\infty(M)$ of infinitely differentiable functions on a smooth manifold M, we get the usual differential-geometric tangent bundle.

Example 1.2. Let (X, R) be a ringed space. Let $\sim$ be an equivalence relation on X, and let $\pi: X \to Y = X/\sim$ be the quotient map modulo $\sim$. Define $f: Y \to \mathbb{R}$ to be admissible iff $f \circ \pi$ is admissible on X. Then we obtain a ringed-space structure on Y.

Example 1.3. Let M be a smooth (C^∞) manifold and let $X \subseteq M$ be any closed subset. Define the admissible functions on X to be all the restrictions to X of C^∞ functions on M. Then this defines a ringed space, $C^\infty(X)$.

Definition 1.4. Let (X, R) be a ringed space, let Y be a topological space, and let $f: Y \to X$ be a continuous mapping. Then Y has an *induced ringed-space structure*: A function on an open subset $U \subseteq Y$ is admissible iff it is locally the pullback of an admissible function on X.

It can be easily checked that we get an embedding $T_pY \subseteq T_{f(p)}X$ at every $p \in Y$.

1.2. Vector fields. In this section we fix a ringed space (X, R). Let $U \subseteq X$ be an open subset.

Definition 1.5. A *vector field* ξ on U is a map $p \mapsto \xi_{|p}$, assigning an element of T_pX to each point $p \in U$. The *derivative* ξf of an admissible function $f: U \to \mathbb{R}$ in the direction of the field ξ, is the function $\xi f: U \to \mathbb{R}$ defined by $(\xi f)(p) = \xi_{|p}\tilde{f}$, where $f \mapsto \tilde{f}$ is the natural quotient map $R(U) \to \mathscr{M}_p/\mathscr{M}_p^2$. We also require that $f \mapsto \xi f$ will map admissible functions into admissible functions.

Definition 1.6. Let ξ, η be admissible vector fields on U. Fix $p \in U$. The map $f \mapsto (\xi\eta f - \eta\xi f)(p)$ defines an element $[\xi, \eta]_p$ of $(\mathscr{M}_p/\mathscr{M}_p^2)^* \cong T_pX$. The assignment $p \mapsto [\xi, \eta]_p$ defines a vector field, which we call the *commutator* $[\xi, \eta]$ of ξ and η.

Definition 1.7. A *2-tensor,* ω, on X is an assignment $p \mapsto \omega_p$, that assigns to each point $p \in X$ a bilinear form ω_p on T_pX, such that $\omega(\xi, \eta)(p) = \omega_p(\xi_{|p}, \eta_{|p})$ is an admissible function on X whenever ξ, η are admissible vector fields.

Definition 1.8. The 2-tensor ω is a *2-form* iff each ω_p is an *antisymmetric* bilinear form on T_pX.

Definition 1.9. A 2-form ω is *closed* iff

$$\underset{\xi, \eta, \zeta}{\text{cyclic sum}} \{\xi\omega(\eta, \zeta) - \omega([\xi, \eta], \zeta)\} = 0$$

for every three vector fields ξ, η, ζ.

Definition 1.10. Let V be a topological vector space. An admissible *V-valued 2-tensor* on X is an assignment $p \mapsto \omega_p$, $\omega_p \in \mathrm{Hom}(T_pX \otimes T_pX, V)$, such that for every continuous linear functional $f \in V^*$, the composition $f \circ \omega_p$ is an admissible real-valued 2-tensor on X.

1.3. **Symplectic structure.**

Definition 1.11. Let B be an antisymmetric bilinear form on the vector space V. B is *degenerate* on the subspace $W \subseteq V$ iff for all $w \in W$ and $v \in V$, $B(w, v) = 0$. The *zero space* of B is the maximal such subspace, i.e., $\{w \in V | B(w, v) = 0 \; \forall v \in V\}$. B is *nondegenerate* iff its zero space is $\{0\}$.

Definition 1.12. Let (X, R) be a ringed space. A *symplectic form* on X is a closed 2-form ω, such that at each $p \in X$, ω_p is a nondegenerate form on T_pX.

Assume that the connected Lie group G acts on the ringed space (X, R) by structure-preserving maps. Further assume that for every fixed $p \in X$, the map $a \mapsto ap$ from G to X is structure preserving, from the ringed space $C^\infty(G)$ to the ringed space (X, R). Denote by O_p the G-orbit of the point $p \in X$. The group G induces a topology on O_p, by identifying $O_p = G/\mathrm{Stab}(p)$. We will put on O_p the ringed-space structure which is induced by the inclusion map $O_p \subseteq X$, as described in Definition 1.4. Then we have an embedding $T_pO_p \subseteq T_pX$.

We denote by X/G the space of all G-orbits in X.

Definition 1.13. A *pre-symplectic form* on (X, R) (relative to the G-action) is a closed 2-form ω on X, which is G-invariant, and such that the zero-space of

ω_p is exactly T_pO_p at each $p \in X$. Such a form is said to define a *symplectic structure* on X/G.

1.4. *Remarks.* Notice that in Definition 1.13 we do not take a tensor on the quotient ringed-space X/G. The reason for this is that the linear quotient space T_pX/T_pO_p is not, in general, the tangent space of X/G: We can define a ringed-space structure on the quotient X/G by taking G-invariant smooth functions on X, as described in Example 1.2. The quotient map $X \to X/G$ will then induce a linear map

$$T_pX/T_pO_p \to T_p(X/G). \tag{5}$$

If $U \subseteq X$ is a G-invariant open set, and if the quotient $U \to U/G$ is a submersion of smooth manifolds, then (5) will be an isomorphism at all the points p in U. At other p's it may not be an isomorphism, though.

Take, for example, the adjoint action of the group $SU(2)$ on itself: $G = X = SU(2)$. An invariant function f on X is determined by its values on the maximal torus $\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{smallmatrix}\right)$. Assume that f vanishes at the identity $p = I$. Invariance under the Weyl group S_2, that switches θ with $-\theta$, forces f to vanish to first order in θ. Therefore, f pulls back to a function on X which is in $\mathscr{M}_I^2$, in the notation of Definition 1.3. So the quotient $X \to X/G$ induces the zero map on the tangent spaces, and the map $T_IX/T_IO_I \cong T_IX \to T_I(X/G)$ is not an isomorphism.

2. Smooth structure on $\mathrm{Hom}(\pi, G)$

2.1. **Smooth structure on product spaces.** Let π be a finitely generated discrete group, and let G be a Lie group. Consider the ringed space with G^π as the underlying topological space and where a function $F: G^\pi \to \mathbb{R}$ is admissible iff it is locally a C^∞ function of a finite number of coordinates. These functions will be called *smooth* on G^π, namely

Definition 2.1. $F: G^\pi \to \mathbb{R}$ is *smooth* iff at each $\phi_0 \in G^\pi$ there is a finite set of coordinates $\{\gamma_1, \dots, \gamma_n, \delta_1, \dots, \delta_k\} \in \pi$ and an open neighborhood U_0 of the unit element in G, such that on the "cylinder" $U = \{\phi: \pi \to G | \phi(\delta_i)\phi_0(\delta_i)^{-1} \in U_0,\ i = 1, \dots, k\}$ one can write $F(\phi) = F_0(\phi(\gamma_1), \dots, \phi(\gamma_n))$ for some C^∞ function $F_0: G^n \to \mathbb{R}$.

Comment 2.1. For a finite subset $I \subseteq \pi$, take the sheaf $C^\infty(G^I)$ of smooth functions on the manifold G^I. The set of such I's is partially ordered by inclusion, and the projection $\Pi^I_J: G^I \to G^J$, for $J \subseteq I$, gives rise to an embedding $\Pi^{I*}_J: C^\infty(G^J) \to C^\infty(G^I)$. This enables one to define: $C^\infty(G^\pi) = \varinjlim_{I \subseteq \pi,\ I \text{ finite}} C^\infty(G^I)$ as a direct limit of *sheaves*, which gives the structure on G^π that was explicitly defined above.

2.2. **Smooth structure on** $\mathrm{Hom}(\pi, G)$. The set of all homomorphisms $\mathrm{Hom}(\pi, G)$ is a subset of the set G^π of all functions from π to G. The inclusion map $\mathrm{Hom}(\pi, G) \to G^\pi$ induces a ringed space structure in $\mathrm{Hom}(\pi, G)$, as described in Definition 1.4. Another way to get a smooth structure on $\mathrm{Hom}(\pi, G)$ is by choosing m generators $\gamma_1, \ldots, \gamma_m$ to π and embedding $\mathrm{Hom}(\pi, G)$ in G^m by $f \mapsto (f(\gamma_1), \ldots, f(\gamma_m))$. This new ringed-space structure is equivalent to the one induced from G^π. Indeed, the following is a commutative diagram of ringed-space morphisms

$$\begin{tikzcd}
 & G^m = \mathrm{Hom}(F_m, G) \arrow[dr] & \\
\mathrm{Hom}(\pi, G) \arrow[ur] \arrow[dr] & & G^{F_m} \\
 & G^\pi \arrow[ur] &
\end{tikzcd}$$

where F_m is a free group over m generators. Moreover, the smooth structures of both G^m and G^π are the ones induced from G^{F_m}, so they induce the same smooth structure on $\mathrm{Hom}(\pi, G)$. Indeed, any function $f: G^m \to \mathbb{R}$ is the pullback of the function $G(\varphi) = f(\varphi(x_1), \ldots, \varphi(x_m))$ on G^{F_m}, with $x_1, \ldots, x_m$ being the generators of F_m. To check the arrow $G^\pi \to G^{F_m}$, take a smooth function $f: G^\pi \to \mathbb{R}$, and fix an open subset $V \subseteq G^\pi$ on which $f(\phi) = F(\phi(\gamma_1), \ldots, \phi(\gamma_n))$, $\phi \in V$, as in Definition 2.1. Fix $P_1, \ldots, P_n \in F_m$ that are mapped to $\gamma_1, \ldots, \gamma_n$ by the quotient $F_m \to \pi$. Then on V, f is the pullback of the function $g(\varphi) = F(\varphi(P_1), \ldots, \varphi(P_n))$, $\varphi \in G^{F_m}$.

Theorem 1. *Every smooth function on* $\mathrm{Hom}(\pi, G)$ *is the restriction of a globally defined smooth function on* G^π.

Proof. Take any function $f: \mathrm{Hom}(\pi, G) \to \mathbb{R}$. Recalling Definition 1.4, we must prove that if f equals *locally* to the restriction of a smooth function on G^π, then it also equals *globally* to the restriction of such a function. This will be proved in two steps.

(1) Let (X, R) be a ringed space that satisfies the following two properties.

(a) X is a paracompact topological space (i.e., each open covering of X has a locally finite refinement).

(b) Each locally finite, open covering of X has a partition of unity subordinate to it, which consists of admissible functions on X.

Let Y be a closed subset of X. Let $\mathscr{U} = \{U_\alpha\}$ be an open covering of X. Let $f: Y \to \mathbb{R}$ and $f_\alpha: U_\alpha \to \mathbb{R}$ be admissible functions, such that $f_\alpha = f$ on $U_\alpha \cap Y$. Then f is the restriction to Y of one admissible function on X.

Proof. By property (1)(a), we may assume that $\mathscr{U}$ is a locally finite covering (or otherwise, use a locally finite refinement of $\mathscr{U}$ and restrictions of the f_α's to the elements of this new covering). By property (1)(b), we have a partition of unity $\{u_\alpha\}$ subordinate to $\mathscr{U}$. Define on $X: F = \sum_\alpha u_\alpha f_\alpha$. Then F is admissible and satisfies $F_{|Y} = f$.

(2) The subset $Y = \mathrm{Hom}(\pi, G)$ is closed in $X = G^\pi$, so it is left to show that G^π satisfies the two properties stated in (1).

Property (1)(a) results from the fact that G^π is metrizable and has an enumerable basis for its topology. As for property (1)(b), we can assume that the covering consists of cylindrical sets $\{V_i\}$. For each V_i we can take a smooth nonnegative function f_i whose zero set is exactly V_i^c, by ignoring all but a finite number of coordinates, defining f_i first on the finite product G^I, $I \subseteq \pi$, and pulling back to G^π. Then all that is left is to normalize the f_i's as to satisfy $\sum f_i = 1$ on G^π, which is possible since this sum is everywhere positive and finite. This ends the proof of Theorem 1.

2.3. **The tangent space to G^π.** We will identify the tangent space T_aG, $a \in G$, with the Lie algebra $g = T_eG$, by identifying $\alpha \in g$ with the tangent vector at $t = 0$ to the curve $t \mapsto \exp(t\alpha)a$ in G. That is, we extend $\alpha \in T_1G$ to a *right* invariant vector field ξ_α on G and take its value at a. Note that we have $\xi_{[\alpha,\beta]} = -[\xi_\alpha, \xi_\beta]$, the ξ's being right (not left) invariant vector fields. This gives rise to a natural isomorphism

$$T_\phi(G^\pi) \cong g^\pi. \tag{6}$$

Given any $\gamma \in \pi$, the γth coordinate of the vector field ξ is the function $\xi(\gamma): G^\pi \to g$ obtained by evaluating ξ at γ, at each point of G^π, via (6). The following theorem can be easily verified.

Theorem 2. *The vector field ξ is smooth* (*as in Definition* 1.5), *if and only if all its coordinates are smooth functions from G^π to g.*

Fixing a function $u: \pi \to g$, we construct a "parallel" vector field ξ_u on G^π, by setting

$$\xi_{u|_\phi} = u \tag{7}$$

at every $\phi \in G^\pi$, using (6). These vector fields satisfy, for $u, v: \pi \to g$,

$$\xi_{[u,v]} = -[\xi_u, \xi_v],$$

where $[u, v]$ is defined as pointwise Lie multiplication. (This is due to the fact that the commutator $[\xi_u, \xi_v]$ can be calculated on each coordinate independently.)

2.4. **The tangent space to** $\mathrm{Hom}(\pi, G)$. $\mathrm{Hom}(\pi, G)$ is the intersection $\bigcap_{y,z\in\pi} X_{y,z}$, where $X_{y,z} = \{\phi: \pi \to G \mid \phi(yz) = \phi(y)\phi(z)\}$. By Definition 1.3 we have [1]

$$T_\phi \mathrm{Hom}(\pi, G) = \bigcap_{y,z\in\pi} T_\phi X_{y,z} \tag{8}$$

$$= \bigcap_{y,z\in\pi} \{u: \pi \to g \mid u(yz) = u(y) + \mathrm{Ad}\,\phi(y)u(z)\}. \tag{9}$$

The second equality above follows from the following calculation in G^3. A tangent vector $\xi \in T_{(a,b,c)}G^3$ with coordinates $(\alpha, \beta, \gamma) \in g^3$, acts on the projection function $(a, b, c) \mapsto a$ by $\xi(a) = \alpha a$ where the a on the left denotes the projection function, and where the right term is right translation of

[1] Careful, see the remark at the end of this section.

$\alpha \in g$ into T_aG. The condition for ξ to be tangent to the subspace $\{F = I\}$, for some $F: G^3 \to G$, is $\xi F = 0$. We now take $F(a, b, c) = abc^{-1}$. By the Leibnitz rule,

$$\begin{aligned}\xi F &= (\xi a)bc^{-1} + a(\xi b)c^{-1} + ab(\xi(c^{-1}))\\ &= (\alpha a)bc^{-1} + a(\beta b)c^{-1} - ab(c^{-1}(\gamma c)c^{-1})\\ &= \alpha + \mathrm{Ad}(a)\beta - \gamma.\end{aligned}$$

Setting $(a, b, c) = (\phi(y), \phi(z), \phi(yz))$, and $\xi = (u(y), u(z), u(yz))$, we get the conditions $u(yz) = u(y) + \mathrm{Ad}\,\phi(y)u(z)$ for $u \in g^\pi$ to be tangent to $\mathrm{Hom}(\pi, G)$. (Computations of a similar flavor appear in [G, §3].)

G acts on G^π by pointwise conjugation. $\mathrm{Hom}(\pi, G)$ is preserved by this action. The tangent at $\phi \in G^\pi$ to its orbit $O_\phi = \{\mathrm{Ad}(a)\phi | a \in G\}$, consists of the vectors $\{u_\gamma,\ \gamma \in g\}$, where $u_\gamma: \pi \to g$ is given by $u_\gamma(x) = \gamma - \mathrm{Ad}\,\phi(x)\gamma$. Indeed, u_γ is the tangent to the curve $\mathrm{Ad}(a_t)\phi$, for $a_t = \exp(t\gamma)$, and it is enough to look only at those vectors which are tangent to curves, since the orbit is a smooth manifold in the usual sense.

Fix $\phi \in \mathrm{Hom}(\pi, G)$. The composition of ϕ with the adjoint action of G on its Lie algebra g, turns g into a π-module, $g_{\mathrm{Ad}\,\phi}$, with $xu = \mathrm{Ad}\,\phi(x)(u)$ for $x \in \pi$ and $u \in g$. Recall the construction of the group cohomology of π with coefficients in the module $g_{\mathrm{Ad}\,\phi}$: The cochains, cocycles, and coboundaries in dimension 1 are, respectively,

$$\begin{aligned}C^1(\pi, g_{\mathrm{Ad}\,\phi}) &= \{\text{functions } \pi \to g_{\mathrm{Ad}\,\phi}\},\\ Z^1(\pi, g_{\mathrm{Ad}\,\phi}) &= \{u: \pi \to g | \forall x, y \in \pi\ u(xy) = u(x) + \mathrm{Ad}\ \phi(x)u(y)\},\\ B^1(\pi, g_{\mathrm{Ad}\,\phi}) &= \{u: \pi \to g | u(x) = \mathrm{Ad}\,\phi(x)\gamma - \gamma \text{ for some fixed } \gamma \in g\}.\end{aligned}$$

To summarize,

Theorem 3. *The tangent space to* G^π, $\mathrm{Hom}(\pi, G)$, *and* O_ϕ, *at the point* $\phi \in \mathrm{Hom}(\pi, G)$ *are, respectively,* $C^1(\pi, g_{\mathrm{Ad}\,\phi})$, $Z^1(\pi, g_{\mathrm{Ad}\,\phi})$, *and* $B^1(\pi, g_{\mathrm{Ad}\,\phi})$.

The dual space $g^*_{\mathrm{Ad}\,\phi}$ is also a π-module. The cup product

$$\cup: C^1(\pi, g^*_{\mathrm{Ad}\,\phi}) \times C^1(\pi, g_{\mathrm{Ad}\,\phi}) \to C^2(\pi, \mathbb{R}) \tag{10}$$

is given by the formula

$$f \cup g(x, y) = \langle f(x), \mathrm{Ad}\,\phi(x)g(y)\rangle$$

and will be used to construct the presymplectic form on $\mathrm{Hom}(\pi, G)$.

Remark. Equations (8) and (9) may fail to be true at the singular points of $\mathrm{Hom}(\pi, G)$. One can correct this by taking a more subtle definition of "smooth structure" (imitating the one in algebraic geometry). The set $\mathscr{H} = \mathrm{Hom}(\pi, G)$ is cut out of G^π by the equations $\{F_{y,z}(\phi) = 1\}_{y,z\in\pi}$ where $F_{y,z}(\phi) = \phi(y \cdot z)\phi(z)^{-1}\phi(y)^{-1}$. Define $C^\infty(\mathscr{H})$ to be the following sheaf. An open set in $\mathscr{H}$ is $U \cap \mathscr{H}$ where $U \subseteq G^\pi$ is open. To this set we associate the quotient ring $C^\infty(U)/(F_{y,z})$. Downstairs, the functions " $(F_{y,z})$ " by which we divide are those of the form $f \circ F_{y,z}$ where $f\colon G \to \mathbb{R}$ is any smooth function that vanishes at $1 \in G$. This is an extension of the ring $\{$*restrictions to* $\mathscr{H}$ *of smooth functions on* $U\}$ which was used earlier. With this new definition of a smooth structure we still have $T_p\mathscr{H} \subseteq T_p(G^\pi)$ $\forall p \in \mathscr{H}$. Theorem 3 then works at all points of $\mathscr{H}$ and so does the rest of this paper.

3. Construction of the symplectic form

Assume that G admits a bilinear form

$$K\colon g \times g \to \mathbb{R} \tag{11}$$

on its Lie algebra, which is symmetric, Ad-invariant, and nondegenerate. (Such a form exists if G is a reductive Lie group.) K induces an isomorphism of the π-modules $g_{\mathrm{Ad}\,\phi} \cong g^*_{\mathrm{Ad}\,\phi}$, whose composition with the cup product (10) is the map

$$\cup\colon C^1(\pi\,,\, g_{\mathrm{Ad}\,\phi}) \times C^1(\pi\,,\, g_{\mathrm{Ad}\,\phi}) \to C^2(\pi\,,\, \mathbb{R})$$

given by

$$u \cup v(c\,,\, d) = K(u(c)\,,\, \mathrm{Ad}\,\phi(c)v(d)). \tag{12}$$

Using this same formula also for ϕ's outside $\mathrm{Hom}(\pi\,,\, G)$, we obtain a 2-tensor ω on G^π, which takes values in the vector space $C^2(\pi\,,\, \mathbb{R}) = \mathbb{R}^{\pi\times\pi}$. At the point $\phi \in G^\pi$, the tangent to G^π is the space g^π, in which the tensor is given by

$$\omega_\phi(u\,,\, v)(c\,,\, d) = K(u(c)\,,\, \mathrm{Ad}\,\phi(c)v(d)) \tag{13}$$

for $u\,,\, v \in g^\pi$ and $(c\,,\, d) \in \pi \times \pi$. In general, this is not the operation of cup product: If ϕ is not a homomorphism, then the Lie algebra g does not turn into a π module, since $\mathrm{Ad}\,\phi\colon \pi \to \mathrm{Aut}(g)$ is not a homomorphism of groups. The properties of ω are summarized below.

Theorem 4. *We have constructed a vector-valued tensor ω on the space G^π, that takes values in the vector space $\mathbf{C}^2(\pi\,,\, \mathbb{R})$. The tensor ω has the following properties.*

(1) *Denote by i the inclusion map of* $\mathrm{Hom}(\pi\,,\, G)$ *into G^π. Then $i^*\omega$ takes values in the subspace $Z^2(\pi\,,\, \mathbb{R})$ of $C^2(\pi\,,\, \mathbb{R})$. Its composition with the quotient map $Z^2(\pi\,,\, \mathbb{R}) \to H^2(\pi\,,\, \mathbb{R})$ is a 2-form on* $\mathrm{Hom}(\pi\,,\, G)$ *which is degenerate along the G-orbits.*

(2) *Let φ be a continuous linear functional, $\varphi\colon Z^2(\pi\,,\, \mathbb{R}) \to \mathbb{R}$, which vanishes on $B^2(\pi\,,\, \mathbb{R})$. Then $\varphi \circ i^*\omega$ is a closed 2-form on* $\mathrm{Hom}(\pi\,,\, G)$, *which is degenerate along the G-orbits. If these are the only degenerate directions, then $\varphi \circ^* \omega$ defines a symplectic structure on* $\mathrm{Hom}(\pi\,,\, G)/G$, *in the sense of Definition* 1.13.

Corollary. *Let $\pi = \pi_1(S)$ be the fundamental group of a compact Riemann surface. S is an Eilenberg-MacLane space, therefore its cohomology is the same as the cohomology of its fundamental group π. The restriction of ω*

to $\mathrm{Hom}(\pi, G)$, *composed with the quotient* $Z^2(\pi, \mathbb{R}) \to H^2(\pi, \mathbb{R})$ *and followed by an isomorphism* $H^2(\pi, \mathbb{R}) \xrightarrow{\cong} \mathbb{R}$, *defines a symplectic structure on* $\mathrm{Hom}(\pi, G)/G$. *The nondegeneracy of the form follows from the Poincaré duality theorem for cohomology with local coefficients. This is the symplectic structure that Goldman constructs in* [G].

Proof. Theorems 2 and 1 imply that every vector field on $\mathrm{Hom}(\pi, G)$ is the restriction of some vector field on G^π. It follows that the restriction of ω to $\mathrm{Hom}(\pi, G)$ is a smooth tensor on $\mathrm{Hom}(\pi, G)$.

The cup product $C^1(\pi, g_{\mathrm{Ad}\,\phi}) \times C^1(\pi, g_{\mathrm{Ad}\,\phi}) \to C^2(\pi, \mathbb{R})$ induces an antisymmetric bilinear form on the cohomology: $H^1(\pi, g_{\mathrm{Ad}\,\phi}) \times H^1(\pi, g_{\mathrm{Ad}\,\phi}) \to H^2(\pi, \mathbb{R})$. By Theorem 3, $T_\phi(\mathrm{Hom}(\pi, G)) = Z^1(\pi, g_{\mathrm{Ad}\,\phi})$ and $T_\phi(O_\phi) = B^1(\pi, g_{\mathrm{Ad}\,\phi})$. Part 1 follows immediately.

The rest of the section is devoted to the proof of part 2.

Define, on G^π:

$$\tilde{\omega}(\xi, \eta) = 1/2(\omega(\xi, \eta) - \omega(\eta, \xi)). \tag{14}$$

This is a 2-form on G^π with values in $C^2(\pi, \mathbb{R})$. Part 1 implies that $\varphi \circ i^*\tilde{\omega} = \varphi \circ i^*\omega$ on $\mathrm{Hom}(\pi, G)$. The cyclic sum in Definition 1.9 defines a 3-form, $d\tilde{\omega}$, on G^π. Then:

$$d(\varphi \circ i^*\omega) = \varphi \circ d i^*\omega = \varphi \circ i^* d\tilde{\omega}.$$

For this to vanish on $\mathrm{Hom}(\pi, G)$ it is enough to show that for any $\Phi \in \mathrm{Hom}(\pi, G)$ and any $u, v, w \in T_\Phi \mathrm{Hom}(\pi, G)$, $(d\tilde{\omega})_{|\Phi}(u, v, w)$ is a coboundary in $C^2(\pi, \mathbb{R})$.

$$(d\tilde{\omega})_{|\Phi}(u, v, w) = \operatorname*{cyclic\ sum}_{\xi, \eta, \zeta} \xi\tilde{\omega}(\eta, \zeta) - \tilde{\omega}([\xi, \eta], \zeta), \tag{15}$$

where ξ, η, ζ are any extensions of u, v, w to vector fields on G^π. We will compute this cyclic sum for the easiest extension we can think of: "parallel" vector fields. So for any $c \in \pi$, the c-coordinate of ξ is a g-valued function on G^π which is constant: $\xi_{|\Phi}(c) = u(c)$ $\forall \Phi \in G^\pi$. Similarly: $\eta_{|\Phi}(c) = v(c)$ and $\zeta_{|\Phi}(c) = w(c)$ $\forall \Phi \in G^\pi$, $\forall c \in \pi$. For computing (15), we will need:

Lemma 5. *Fix* $\alpha, \beta \in g$ *and consider the real valued function on* G^π *given by* $\Phi \mapsto K(\alpha, \mathrm{Ad}\,\Phi(c)\beta)$. *Then its Lie derivative in the direction of the vector field* ξ *is:* $K(\alpha, [\xi_{|\Phi}(c), \mathrm{Ad}\,\Phi(c)\beta])$.

Proof. By the bilinearity of K, we need only to look at the function $a \mapsto \mathrm{Ad}(a)\beta$ from G to g, and to show that its derivative along a "parallel" vector field $\xi = \xi_\gamma$ $(\gamma \in g)$ is the function $a \mapsto [\gamma, \mathrm{Ad}(a)\beta]$. Recall that ξ_γ is the right invariant vector field on G whose value at the origin is γ. For $\xi = \xi_\gamma$,

$$\begin{aligned}\xi(\mathrm{Ad}(a)\beta) &= \xi(a\beta a^{-1}) = (\xi a)\beta a^{-1} + a\beta(-a^{-1}\xi(a)a^{-1}) \\ &= (\gamma a)\beta a^{-1} - a\beta a^{-1}(\gamma a)a^{-1} = [\gamma, \mathrm{Ad}(a)\beta].\end{aligned}$$

Now let us bravely compute (15). To save parentheses, denote the c-coordinate of any vector field ξ by ξ_c $(c \in \pi,\ \xi_c: G^\pi \to g)$. Then

$$\begin{aligned}
&2d\tilde{\omega}(\xi, \eta, \zeta)(c, d) \\
&\quad = \underset{\xi,\eta,\zeta}{\text{cyclic sum}}\ \xi\omega(\eta, \zeta)(c, d) - \omega([\xi, \eta], \zeta)(c, d) \\
&\qquad\qquad - \xi\omega(\zeta, \eta)(c, d) + \omega(\zeta, [\xi, \eta])(c, d) \\
&\quad = \underset{\xi,\eta,\zeta}{\text{cyclic sum}}\ \xi K(\eta_c, \operatorname{Ad}\Phi(c)\zeta_d) - K([\xi, \eta]_c, \operatorname{Ad}\Phi(c)\zeta_d) \\
&\qquad\qquad - \xi K(\zeta_c, \operatorname{Ad}\Phi(c)\eta_d) + K(\zeta_c, \operatorname{Ad}\Phi(c)[\xi, \eta]_d) \\
&\quad = \underset{\xi,\eta,\zeta}{\text{cyclic sum}}\ K(\eta_c, [\xi_c, \operatorname{Ad}\Phi(c)\zeta_d]) + K([\xi_c, \eta_c], \operatorname{Ad}\Phi(c)\zeta_d) \\
&\qquad\qquad - K(\zeta_c, [\xi_c, \operatorname{Ad}\Phi(c)\eta_d]) - K(\zeta_c, \operatorname{Ad}\Phi(c)[\xi_d, \eta_d]) \\
&\quad = \underset{\xi,\eta,\zeta}{\text{cyclic sum}}\ - K([\xi_c, \eta_c], \operatorname{Ad}\Phi(c)\zeta_d) + K([\xi_c, \eta_c], \operatorname{Ad}\Phi(c)\zeta_d) \\
&\qquad\qquad + K([\xi_c, \zeta_c], \operatorname{Ad}\Phi(c)\eta_d) - K(\zeta_c, \operatorname{Ad}\Phi(c)[\xi_d, \eta_d]) \\
&\quad = -\underset{\xi,\eta,\zeta}{\text{cyclic sum}}\ K(\operatorname{Ad}\Phi(c)\xi_d, [\eta_c, \zeta_c]) + K(\xi_c, \operatorname{Ad}\Phi(c)[\eta_d, \zeta_d]).
\end{aligned}$$

The first equality above follows from (14) and Definition 1.9. The second equality follows from (13). The third follows from Lemma 5 and from $[\xi, \eta]_c = -[\xi_c, \eta_c]$. The fourth follows from the Ad-invariance of K. The last equality is a permutation of the cyclic sum.

To finish, we want to show that the function that we are left with, namely

$$F(c, d) = \underset{\xi,\eta,\zeta}{\text{cyclic sum}}\ K(\operatorname{Ad}\Phi(c)\xi_d, [\eta_c, \zeta_c]) + K(\xi_c, \operatorname{Ad}\Phi(c)[\eta_d, \zeta_d]),$$

is a coboundary in $C^2(\pi, \mathbb{R})$, i.e., that it equals $f(cd) - f(c) - f(d)$ for some $f: \pi \to \mathbb{R}$. We will show that the following function f does the job:

$$f(c) = K(\xi_c, [\eta_c, \zeta_c]).$$

Remember that

$$\xi_{cd} = \xi_c + \operatorname{Ad}\Phi(c)\xi_d, \tag{16}$$

$$\eta_{cd} = \eta_c + \operatorname{Ad}\Phi(c)\eta_d, \tag{17}$$

$$\zeta_{cd} = \zeta_c + \operatorname{Ad}\Phi(c)\zeta_d \tag{18}$$

for any $c, d \in \pi$, since ξ, η, ζ are parallel extensions of u, v, w which are in $Z^2(\pi, g_{\operatorname{Ad}\phi})$, being tangent to $\operatorname{Hom}(\pi, G)$ at Φ. So:

$$\begin{aligned}
f(cd) &= K(\xi_{cd}, [\eta_{cd}, \zeta_{cd}]) \\
&= K(\xi_c + \operatorname{Ad}\Phi(c)\xi_d, [\eta_c + \operatorname{Ad}\Phi(c)\eta_d, \zeta_c + \operatorname{Ad}\Phi(c)\zeta_d]).
\end{aligned}$$

This decomposes into eight terms, treated below. The following manipulations only involve the symmetry of K and its Ad-invariance:

$$\begin{aligned}
\text{first term} &= K(\xi_c, [\eta_c, \zeta_c]) = f(c), \\
\text{second term} &= K(\xi_c, [\eta_c, \operatorname{Ad}\Phi(c)\zeta_d]) = K(\operatorname{Ad}\Phi(c)\zeta_d, [\xi_c, \eta_c]), \\
\text{third term} &= K(\xi_c, [\operatorname{Ad}\Phi(c)\eta_d, \zeta_c]) = K(\operatorname{Ad}\Phi(c)\eta_d, [\zeta_c, \xi_c]), \\
\text{fourth term} &= K(\xi_c, \operatorname{Ad}\Phi(c)[\eta_d, \zeta_d]), \\
\text{fifth term} &= K(\operatorname{Ad}\Phi(c)\xi_d, [\eta_c, \zeta_c]), \\
\text{sixth term} &= K(\operatorname{Ad}\Phi(c)\xi_d, [\eta_c, \operatorname{Ad}\Phi(c)\zeta_d]) = K(\eta_c, \operatorname{Ad}\Phi(c)[\zeta_d, \xi_d]), \\
\text{seventh term} &= K(\operatorname{Ad}\Phi(c)\xi_d, [\operatorname{Ad}\Phi(c)\eta_d, \zeta_c]) = K(\zeta_c, \operatorname{Ad}\Phi(c)[\xi_d, \eta_d]), \\
\text{eighth term} &= K(\operatorname{Ad}\Phi(c)\xi_d, \operatorname{Ad}\Phi(c)[\eta_d, \zeta_d]) = K(\xi_d, [\eta_d, \zeta_d]) = f(d).
\end{aligned}$$

So:

$$\begin{aligned}
(\text{fifth} + \text{third} + \text{second terms}) &= \underset{\xi, \eta, \zeta}{\text{cyclic sum}}\, K(\operatorname{Ad}\Phi(c)\xi_d, [\eta_c, \zeta_c]), \\
(\text{fourth} + \text{sixth} + \text{seventh terms}) &= \underset{\xi, \eta, \zeta}{\text{cyclic sum}}\, K(\xi_c, \operatorname{Ad}\Phi(c)[\eta_d, \zeta_d]), \\
(\text{first} + \text{eighth terms}) &= f(c) + f(d).
\end{aligned}$$

So:

$$\begin{aligned}
f(cd) - f(c) - f(d) = \underset{\xi, \eta, \zeta}{\text{cyclic sum}}\, & K(\operatorname{Ad}\Phi(c)\xi_d, [\eta_c, \zeta_c]) \\
& + K(\xi_c, \operatorname{Ad}\Phi(c)[\eta_d, \zeta_d])
\end{aligned}$$

as desired.

4. Application

In this section we will use Theorem 4 to show that Goldman's construction works in the following case.

Theorem 5. *Let X be a compact Kähler manifold. Let π be its fundamental group. Then the space* $\operatorname{Hom}(\pi, G)/G$ *admits a symplectic structure.*

Proof. By Theorem 4, all we need is a linear map $\varphi: H^2(\pi, \mathbb{R}) \to \mathbb{R}$, whose composition with the cup product gives a nondegenerate pairing $H^1(\pi, g_{\operatorname{Ad}\phi}) \times H^1(\pi, g_{\operatorname{Ad}\phi}) \to \mathbb{R}$.

In constructing φ, we will use several equivalent definitions for cohomology with local coefficients. First, for the de Rham setup, fix a flat vector bundle $E \to X$ whose fiber is our π-module $g_{\operatorname{Ad}\phi}$ and whose holonomy is given by the action of π on this module. A differential form of degree k on X, with coefficients in E, is a section of $\bigwedge^k T^*X \otimes E$. Locally it can be written as $\sum_j \alpha_j \otimes e_j$ where α_j are k-forms on X and e_j are flat sections of E. The exterior derivative acts by $d(\sum_j \alpha_j \otimes e_j) = \sum_j d\alpha_j \otimes e_j$. This gives a complex whose cohomology $H^*_{DR}(X, E)$ is, by definition, the de Rham cohomology of X with coefficients in the bundle E. (See [BT, p. 79].) These groups are strongly related to the cohomology of the group π with values in the module $g_{\operatorname{Ad}\phi}$: The groups $H^1_{DR}(X, E)$ and $H^1(\pi, g_{\operatorname{Ad}\phi})$ are isomorphic, and if $\pi_2(X) = 0$, then $H^2_{DR}(X, E)$ and $H^2(\pi, g_{\operatorname{Ad}\phi})$ are isomorphic too. In fact, if X is aspherical (i.e., $\pi_i(X) = 0$ for $i \geq 2$) then $H^k_{DR}(X, E)$ and $H^k(\pi, g_{\operatorname{Ad}\phi})$ are isomorphic for all k.

This construction works for any flat bundle over X. If we take the trivial bundle with fiber $\mathbb{R}$, then we get the standard de Rham cohomology of X, and if $\pi_2(X) = 0$ then again $H^2_{DR}(X, \mathbb{R}) \cong H^2(\pi, \mathbb{R})$. The Ad-invariant bilinear form (11) defines a scalar product on the fibers of our vector bundle. The cup product of two forms is then defined by their wedge product, where the coefficients are multiplied via the scalar product on the fibers. The result is a differential form on X with coefficients in $\mathbb{R}$:

$$(\alpha_1 \otimes s_1) \wedge (\alpha_2 \otimes s_2) = \langle s_1, s_2 \rangle \alpha_1 \wedge \alpha_2.$$

Poincaré duality applies here. In particular, if $\dim_{\mathbb{R}} X = n$, then we get a nondegenerate pairing of the cohomology

$$H^1_{DR}(X, E) \times H^{n-1}_{DR}(X, E) \xrightarrow{\text{cup product}} H^n_{DR}(X, \mathbb{R}). \tag{19}$$

Let ω be the symplectic form on X (i.e., the imaginary part of the Kähler metric), and let $[\omega]$ be its cohomology class in $H^2_{DR}(X, \mathbb{R})$. The Hard Lefschetz Theorem [GH, Chapter 0, §7] applies here too. It says that the map $[\alpha] \mapsto [\alpha] \cup [\omega]^{n/2-1}$ is an isomorphism $H^1_{DR}(X, E) \xrightarrow{\cong} H^{n-1}_{DR}(X, E)$. Composing this with (19) gives a nondegenerate pairing

$$H^1_{DR}(X, E) \times H^1_{DR}(X, E) \to H^n_{DR}(X, \mathbb{R}).$$

This the same as first taking the cup product $H^1_{DR}(X, E) \times H^1_{DR}(X, E) \to H^2_{DR}(X, \mathbb{R})$ and then composing it with the map

$$[\alpha] \mapsto [\alpha] \cup [\omega]^{n/2-1} \tag{20}$$

from $H^2_{DR}(X, \mathbb{R})$ to $H^n_{DR}(X, \mathbb{R})$. Composing this further with an isomorphism

$$H^n_{DR}(X, \mathbb{R}) \xrightarrow{\cong} \mathbb{R}, \tag{21}$$

gives a nondegenerate pairing

$$H^1_{DR}(X, E) \times H^1_{DR}(X, E) \to \mathbb{R}.$$

Using the isomorphism $H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\,\phi}) \cong H^1_{DR}(X, E)$, we get a nondegenerate pairing on $H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\,\phi})$, namely

$$\begin{array}{ccccc} H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\,\phi}) \times H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\,\phi}) & & & & \\ \downarrow \qquad\qquad \downarrow & & & & \\ H^1_{DR}(X, E) \times H^1_{DR}(X, E) & \xrightarrow{\text{cup product}} & H^2_{DR}(X, \mathbb{R}) & \xrightarrow{\varphi'} & \mathbb{R} \end{array}$$

where φ' is the composition of (20) with (21).

To finish, we need to show that this pairing splits through the cup product on $H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\,\phi})$. It is enough to find a map F that makes the following diagram commute:

$$\begin{array}{ccccc} H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\,\phi}) \times H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\,\phi}) & \xrightarrow{\text{cup product}} & H^2(\pi, \mathbb{R}) & & \\ \downarrow \qquad\qquad \downarrow & & \Big\downarrow{\scriptstyle F} & & \\ H^1_{DR}(X, E) \times H^1_{DR}(X, E) & \xrightarrow{\text{cup product}} & H^2_{DR}(X, \mathbb{R}) & \xrightarrow{\varphi'} & \mathbb{R} \end{array} \tag{22}$$

The composition of F with φ' is then a map $\varphi: H^2(\pi, \mathbb{R}) \to \mathbb{R}$ which satisfies the nondegeneracy condition for Theorem 4.

If we had $\pi_2(X) = 0$, then we would have an F which is an isomorphism. Otherwise, to construct F we will pass to yet another version of cohomology, namely, singular cohomology with coefficients in a flat vector bundle $E \to X$. A cochain here maps every singular simplex $\sigma: \Delta \to X$ (where Δ is a standard simplex) to a flat section of $\sigma^{-1}E$ over Δ. Here $\sigma^{-1}E$ is the pullback of the bundle $E \to X$ to a flat bundle over Δ:

$$\begin{array}{ccc} \sigma^{-1}E & \to & E \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{\sigma} & X \end{array}$$

The map d is given by the same formula as in the standard singular cohomology. (This is well defined since the bundles $\sigma^{-1}E$ are trivial.) If the base space X is a differential manifold, then the singular cohomology with local coefficients is equivalent to the corresponding de Rham cohomology

$$H^n_{\text{sing}}(X, E) \cong H^n_{DR}(X, E).$$

Now, let $X \to \widetilde{X}$ be an embedding of X in a topological space $\widetilde{X}$ for which $\pi_1(\widetilde{X}) = \pi_1(X) = \pi$ and $\pi_2(\widetilde{X}) = 0$. This can be achieved by gluing to X 3-cells that kill all the generators of $\pi_2(X)$. E extends to a flat bundle $\widetilde{E} \to \widetilde{X}$, such that the following diagram commutes:

$$\begin{array}{ccc} E & \to & \widetilde{E} \\ \downarrow & & \downarrow \\ X & \to & \widetilde{X} \end{array} \tag{23}$$

Indeed, every flat vector bundle over S^2 is trivial. Therefore, for every 3-cell D^3 that we attached to X, it is possible to extend E from a flat bundle over the boundary of D^3 to a flat bundle over all of D^3. Now, via the maps (23), we can pullback the cocycles and thus get a morphism

$$H^n_{\text{sing}}(\widetilde{X}, \widetilde{E}) \to H^n_{\text{sing}}(X, E) \tag{24}$$

for all n. Since $\pi_1(\widetilde{X}) = \pi$, we have

$$H^1_{\text{sing}}(\widetilde{X}, \widetilde{E}) \cong H^1(\pi, g_{\text{Ad}\,\phi}) \tag{25}$$

and since $\pi_2(\widetilde{X}) = 0$, we have

$$H^2_{\text{sing}}(\widetilde{X}, \mathbb{R}) \cong H^2(\pi, \mathbb{R}). \tag{26}$$

(For the second isomorphism, we take a trivial bundle over X with fiber $\mathbb{R}$, rather than our bundle E with fiber g.) The morphisms (24) respect the cup-product operations. Using the isomorphisms (25) and (26), we then get a commuting diagram

$$\begin{array}{ccccc} H^1(\pi, g_{\text{Ad}\,\phi}) & \times & H^1(\pi, g_{\text{Ad}\,\phi}) & \xrightarrow{\text{cup product}} & H^2(\pi, \mathbb{R}) \\ \downarrow & & \downarrow & & \downarrow \\ H^1_{\text{sing}}(X, E) & \times & H^1_{\text{sing}}(X, E) & \xrightarrow{\text{cup product}} & H^2_{\text{sing}}(X, \mathbb{R}) \end{array} \tag{27}$$

Finally, since X is a differentiable manifold, its singular cohomology is isomorphic to its de Rham cohomology. Composing (27) with this isomorphism gives (22), which ends the proof.

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