PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 116, Number 3, November 1992

ON AN EXAMPLE OF AHERN AND RUDIN

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(Communicated by Clifford J. Earle, Jr.)

ABSTRACT. We show that the polynomial hull of a certain totally real three-sphere in $\mathbb{C}^3$ constructed by Ahern and Rudin is the union of a two-parameter family of analytic disks.

1. INTRODUCTION

If M is a compact real n-dimensional submanifold of $\mathbb{C}^n$ then M must have nontrivial polynomial hull $\widehat{M} = \{z \in \mathbb{C}^n: |P(z)| \leq \max_M |P| \text{ for all polynomials } P\}$; a result of Alexander [3] states that the topological dimension of $\widehat{M}\backslash M$ is at least $n+1$. One would like to understand $\widehat{M}$, perhaps by finding analytic disks with boundaries in M (an analytic disk Δ is the image of an analytic map $f: \{z \in \mathbb{C}: |z| < 1\} \to \mathbb{C}^n$; if f extends continuously to $\{|z| \leq 1\}$, and if $\partial\Delta = f(|z| = 1) \subset M$, then by the maximum principle $\Delta \subset \widehat{M}$). If $n = 2$ and M has a complex tangent at $p \in M$, i.e., the tangent space to M at p has a nonzero complex subspace, then generically either the tangent is of hyperbolic type and M is locally polynomially convex, or the tangent is of elliptic type, in which case a technique of Bishop [6] can be used to construct analytic disks with boundaries in M near p. In the same paper, Bishop showed that for any $M \subset \mathbb{C}^2$ diffeomorphic to the two-sphere there are at least two points of compex tangency. However, a theorem of Gromov [9, p. 193] guarantees the existence of embedded three-spheres in $\mathbb{C}^3$ that are totally real (no complex tangents). In this case, different methods must be used to exhibit analytic structure in $\widehat{M}\backslash M$. Ahern and Rudin [1] gave the first explicit example of such a sphere, as a graph over the boundary of the ball in $\mathbb{C}^2$,

$$M = \{z_1, z_2, \overline{z}_1 z_2 \overline{z}_2^2 + i z_1 \overline{z}_1^2 \overline{z}_2): |z_1|^2 + |z_2|^2 = 1\}. \tag{1.1}$$

They did not address the question of determining $\widehat{M}$. Recently Forstneric [7] constructed a totally real three-sphere in $\mathbb{C}^3$ for which he was able to produce a one-parameter family of analytic disks in $\widehat{M}\backslash M$.

The goal of this paper is to show that polynomial hull of the Ahern-Rudin example is foliated by analytic disks.

Received by the editors January 21, 1991; presented at the 865th AMS Meeting, Tampa, FL, March 22, 1991.

1991 *Mathematics Subject Classification.* Primary 32E20.

Theorem 1. *Let M be the totally real three-sphere given by* (1.1). *There exists a two-parameter family $\Delta_{\theta_1, \theta_2}$ of analytic disks such that*

$$\widehat{M}\backslash M = \left\{ \bigcup_{\theta_1, \theta_2} \Delta_{\theta_1, \theta_2} \right\} \cup \Delta_1 \cup \Delta_2$$

where $\Delta_1 = \{|z_1| < 1, \ z_2 = z_3 = 0\}$ *and* $\Delta_2 = \{|z_2| < 1, \ z_1 = z_3 = 0\}$.

The proof uses a result of Wermer that seems to be particularly useful for constructing analytic disks when M is invariant under certain group actions (see also [5, 8]). The arguments used here parallel closely those in [10], where disks in $\mathbb{C}^2$ invariant under $(z, w) \to (ze^{i\theta}, we^{-i\theta})$ are studied.

In related work, Alexander [4] has proved that if M is a graph in $\mathbb{C}^3$ of a function f continuous on the boundary of the ball B in $\mathbb{C}^2$ then $\widehat{M}$ covers B_2; i.e., the projection of $\widehat{M}$ to $\mathbb{C}^2$ is $\overline{B}_2$. He also gives conditions on f under which $\widehat{M}$ is itself a graph over B_2, but his results do not apply to the case considered here. Finally, Ahern and Rudin [2], by different methods than those employed here, have recently generalized our result by describing the hull of a totally real three-sphere M in $\mathbb{C}^3$ of the form $\{(z, w, \Gamma(z\overline{z})/zw) : (z, w) \in bB\}$ where Γ belongs to a certain class of plane curves. In particular, they show that $\widehat{M}\backslash M$ is both a graph over B and a union of analytic disks.

2. Proof of Theorem 1

We note that M is invariant under the transformation

$$z = (z_1, z_2, z_3) \to T_{\theta_1, \theta_2}(z) = (z_1 e^{i\theta_1}, z_2 e^{i\theta_2}, z_3 e^{-i(\theta_1+\theta_2)}).$$

Clearly $\widehat{M}$ must also be T_{θ_1, θ_2}-invariant. Let $F(z) = z_1 z_2 z_3$. Then $F \circ T_{\theta_1, \theta_2} = F$. If $z \in M$ then $F(z) = |z_1|^2 |z_2|^2 (|z_2|^2 + i|z_1|^2)$. Let γ be the image of M under F. Setting $|z_1| = r$, we obtain a parametrization of γ: if

$$\gamma(r) = r^2(1 - r^2)(1 - r^2 + ir^2) \tag{2.1}$$

then $\gamma = \{\gamma(r) : 0 \leq r \leq 1\}$. γ is a simple closed analytic curve in the complex plane, with a double point at the origin and no other singularity. Let Ω be the region bounded by γ. If $\zeta \in \gamma\backslash\{0\}$ then it is easy to check that there exists a unique $r \in (0, 1)$ so that $\gamma(r) = \zeta$. This r we denote by $r(\zeta)$. If $z \in M$ and $F(z) = \zeta \neq 0$, then the orbit of z under the transformations T_{θ_1, θ_2} is the torus $\{T_{\theta_1, \theta_2}(z_r) : 0 \leq \theta_1, \theta_2 < 2\pi\}$ where $z_r = (r, \sqrt{1 - r^2}, \gamma(r)/r\sqrt{1 - r^2})$ and $r = r(\zeta)$. If $F(z) = 0$ then $z_1 = 0$ or $z_2 = 0$ and the orbit of z is the circle $z_1 = z_3 = 0$, $|z_2| = 1$ in the first case, or $z_2 = z_3 = 0$, $|z_1| = 1$ in the second case.

Lemma 1. $F(\widehat{M}) = \overline{\Omega}$.

Proof. Clearly $F(\widehat{M}) \subset \overline{\Omega}$. If $\zeta_0 \in \Omega$ and $\zeta_0 \notin F(\widehat{M})$, then $h(z) = (F(z) - \zeta_0)^{-1}$ is an element of $P(M)$. If $g = \sum_\alpha A_\alpha z^\alpha$ $(\alpha = (\alpha_1, \alpha_2, \alpha_3))$ is any polynomial in (z_1, z_2, z_3), then there exists a polynomial $\tilde{g}$ in one variable so that for $0 < r < 1$,

$$\tilde{g}(\gamma(r)) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} g \circ T_{\theta_1, \theta_2}(z_r)\, d\theta_1\, d\theta_2.$$

In fact we can take $\tilde{g}(\zeta) = \sum a_j \zeta^j$ where $a_j = A_{(j,j,j)}$. Since h is constant on the orbit of z_r, for any polynomial g we can write

$$\tilde{g}(\gamma(r)) - (\gamma(r) - \zeta_0)^{-1} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} ((g - h) \circ T_{\theta_1, \theta_2})(z_r) \theta_1 \, d\theta_2$$

and thus

$$\max_{0 \le r \le 1} |\tilde{g}(\gamma(r)) - (\gamma(r) - \zeta_0)^{-1}| \le \max_{z \in M} |g(z) - h(z)|$$

(extending the inequality by continuity to $r = 0$). Thus if $h \in P(M)$, $(\gamma(r) - \zeta_0)^{-1}$ is uniformly approximable by polynomias on γ. It follows that $(\zeta - \zeta_0)^{-1}$ is uniformly approximable by polynomials in ζ on Ω, which is false. The proof is complete.

Now we use a theorem of Wermer [10]: let A be a uniform algebra on a compact Hausdorff space X, and let M_A denote the maximal ideal space of A. For $f \in A$, let $\hat{f}$ denote the Gelfand transform of f. Define for $g \in A$ and $\zeta \in \hat{f}(M_A)$,

$$\Psi(\zeta) = \log(\max\{|g(p)| : p \in M_A \text{ and } \hat{f}(p) = \zeta\}).$$

Then Wermer's Theorem states that Ψ is subharmonic on $\mathbb{C} \backslash \hat{f}(M_A)$. If for $\zeta \in \Omega$ we set

$$Z_i(\zeta) = \max\{|z_i| : z \in \widehat{K} \text{ and } F(z) = \zeta\}, \qquad i = 1, 2, 3,$$

this result applied to the algebra $A = P(M)$ of uniform limits of polynomials on M implies that $\log Z_i$ is subharmonic on Ω. We will construct analytic disks in $\widehat{M}$ using the functions $\log Z_i$, which we will show are actually harmonic in Ω. First we must examine the boundary behavior of the Z_i. We need

Lemma 2. *Suppose* $z^0 \in M$ *and* $F(z^0) = \zeta^0 \in \gamma \backslash \{0\}$. *Then* $z^0 \in M$.

Proof. Assume $z^0 \notin M$. Let $r \in (0, 1)$ such that $F(z^0) = \gamma(r)$. Let T be the orbit of z_r under the T_{θ_1, θ_2}. T is easily seen to be polynomially convex. Since $z^0 \notin T$, there exists a polynomial P with $|P(z^0)| > 1$, $|P| < 1$ on T. Choose a neighborhood U of T in M with $|P| < 1$ on U. The image of $M \backslash U$ under F is a closed subarc $\tilde{\gamma}$ of γ exlcuding ζ^0. There exists g analytic on Ω, continuous on $\overline{\Omega}$, and $\delta > 0$, so that $g(\zeta^0) = 1$ and $|g| < 1 - \delta$ on $\tilde{\gamma}$. For any n, $Q(z) = (g \circ F(z))^n P(z)$ is a uniform limit of polynomials on $M \cup \{z^0\}$. On U, $|Q| \le |P| < 1$, while on $M \backslash U$, $|Q| < (1 - \delta)^n |P| < 1$ for sufficiently large n. But $|Q(z^0)| = |P(z^0) > 1$, which contradicts $z^0 \in \widehat{M}$ and completes the proof.

Next we use Lemma 2 to show that Z_1 continuously assumes the boundary values $r(\zeta)$ on $\gamma \backslash \{0\}$. Let $\{\zeta_n\}$ be a sequence in Ω with $\zeta_n \to \zeta \in \gamma \backslash \{0\}$. Choose $z^{(n)} \in \widehat{M}$ with $F(z^{(n)}) = \zeta_n$ and $|z_1^{(n)}| = Z_1(\zeta_n)$. We can assume $z^{(n)}$ converges to $z \in \widehat{M}$. Then $F(z) = \zeta$. By Lemma 2, $z \in M$. Thus $Z_1(\zeta_n) \to |z_1| = r(\zeta)$. Similarly we can show that Z_2 and Z_3 continuously assume the boundary values $\sqrt{1 - r^2}$ and $\gamma(r)/r\sqrt{1 - r^2}$, respectively, on $\gamma \backslash \{0\}$. Now we need a regularity result on solutions of the Dirichlet problem on Ω when the boundary data satisfies certain estimates. The proof of the following lemma is essentially contained in the proof of Lemma 4 of [10] and the discussion preceding it.

Lemma 3. *Let* G *be subharmonic on* Ω, *and suppose* $g(\zeta) = \lim_{\zeta' \to \zeta} G(\zeta')$ *exists and is a continuous function of* ζ *on* $\gamma \backslash \{0\}$. *Furthermore suppose there exist positive constants* c, C *so that* $C > g(\zeta) > c \log|\zeta|$ *for all* $\zeta \in \gamma\backslash\{0\}$. *For* $\zeta_0 \in \Omega$ *set*

$$\widetilde{G}(\zeta_0) = \int g(\zeta)\, d\mu_{\zeta_0}(\zeta)$$

where $d\mu_{\zeta_0}$ *is harmonic measure at* ζ_0 *with respect to* Ω. *Then* $\widetilde{G}$ *is harmonic and bounded above on* Ω, *assumes continuously the boundary values* g *on* $\gamma\backslash\{0\}$, *and* $G(\zeta) \le \widetilde{G}(\zeta)$, *for all* $\zeta \in \Omega$.

Now define

$$\begin{aligned} U_1(\zeta_0) &= \int \log r(\zeta)\, d\mu_{\zeta_0}(\zeta), \\ U_2(\zeta_0) &= \int \log(\sqrt{1 - r^2(\zeta)})\, d\mu_{\zeta_0}(\zeta), \\ U_3(\zeta_0) &= \int \log|\zeta| - \log r(\zeta) - \log(\sqrt{1-r^2(\zeta)})\, d\mu_{\zeta_0}(\zeta) \\ &= \log|\zeta_0| - U_1(\zeta_0) - U_2(\zeta_0). \end{aligned}$$

Apply Lemma 3 with $(G, g) = (\log Z_1, \log r)$, $(\log Z_2, \log(\sqrt{1-r^2})$, and $(\log Z_3, \log|\zeta| - \log r(\zeta) - \log(\sqrt{1-r^2(\zeta)})$ in turn. In each case the estimate required on g follows from the fact that $|\zeta| = |\gamma(r(\zeta))| \le r(\zeta)^2(1 - r(\zeta)^2)$. We obtain

$$\begin{aligned} &\log Z_1(\zeta) \le U_1(\zeta), \\ &\log Z_2(\zeta) \le U_2(\zeta), \\ &\log Z_3(\zeta) \le \log|\zeta| - U_1(\zeta) - U_2(\zeta), \end{aligned} \tag{2.2}$$

where the functions on the right-hand side are harmonic on Ω.

Lemma 4. *For all* $\zeta \in \Omega$, $\log Z_1(\zeta) = U_1(\zeta)$, $\log Z_2(\zeta) = U_2(\zeta)$, *and* $\log Z_3(\zeta) = \log|\zeta| - U_1(\zeta) - U_2(\zeta)$.

Proof. If any one of these three inequalities should fail then, by (2.2),

$$\log Z_1(\zeta) + \log Z_2(\zeta) + \log Z_3(\zeta) < \log|\zeta|.$$

But clearly $Z_1(\zeta)Z_2(\zeta)Z_3(\zeta) \ge |\zeta|$, so we have a contradiction.

Now let $V_i(\zeta)$ be the harmonic conjugate in Ω of $U_i(\zeta)$, $i = 1, 2$. Set

$$\varphi_i = e^{U_i + \sqrt{-1} V_i} \qquad i = 1, 2.$$

Then each φ_i is analytic and nonvanishing on Ω. Set

$$\phi(\zeta) = \left(\varphi_1(\zeta), \varphi_2(\zeta), \frac{\zeta}{\varphi_1(\zeta)\varphi_2(\zeta)}\right)$$

ϕ maps Ω analytically into $\mathbb{C}^3$. We claim $\phi(\Omega) \subset \widehat{M}$. Fix $\zeta \in \Omega$. Note $F(\phi(\zeta)) = \zeta$ and, by Lemma 4, $|\varphi_1(\zeta)| = e^{U_1(\zeta)} = Z_1(\zeta)$. Choose $z \in \widehat{M}$ with $F(z) = \zeta$ and $|z_1| = Z_1(\zeta)$. Then $\varphi_1(\zeta) = e^{i\theta_1} z_1$ for some $\theta_1 \in [0, 2\pi)$. If $\log|z_2| < U_2(\zeta)$ then

$$\begin{aligned} \log|\zeta| &= \log|z_1| + \log|z_2| + \log|z_3| \\ &< U_1(\zeta) + U_2(\zeta) + \log|z_3| = \log|\zeta| - \log Z_3(\zeta) + \log|z_3| \end{aligned}$$

by Lemma 4. But then $\log Z_3(\zeta) < \log|z_3|$, a contradiction. Thus $\log|z_2| = U_2(\zeta)$, which implies $|z_2| = |\varphi_2(\zeta)|$, and so $z_2 = \varphi_2(\zeta)e^{i\theta_2}$ for some θ_2. Finally,

$$z_3 = \frac{\zeta}{z_1 z_2} = \frac{\zeta e^{-i(\theta_1+\theta_2)}}{\varphi_1(\zeta)\varphi_2(\zeta)}$$

and so $\phi(\zeta) = T_{\theta_1,\theta_2}(z) \in \widehat{M}$.

Define for $(\theta_1, \theta_2) \in [0, 2\pi) \times [0, 2\pi)$ the analytic disk $\Delta_{\theta_1,\theta_2}$ to be the image of Ω under $T_{\theta_1,\theta_2} \circ \phi$. It is easy to verify that $\Delta_{\theta_1,\theta_2}$ and $\Delta_{\theta_1',\theta_2'}$ are disjoint unless $(\theta_1, \theta_2) = (\theta_1', \theta_2')$. We have shown that $\Delta_{0,0} \subset \widehat{M}$, so $\Delta_{\theta_1,\theta_2}$ is a two-parameter family of disks lying in $\widehat{M}$. Let Δ_1, Δ_2 be as in the statement of Theorem 1. Since $\partial\Delta_1$ and $\partial\Delta_2$ are contained in M, $\Delta_1 \cup \Delta_2 \subset \widehat{M}$. To complete the proof of Theorem 1 we must show that each $z \in \widehat{M}\backslash M$ is contained in some $\Delta_{\theta_1,\theta_2}$ or in $\Delta_1 \cup \Delta_2$. By Lemma 2, either $F(z) = \zeta \in \Omega$, or $F(z) = 0$. In the first case, we must have $|z_i| = Z_i(\zeta)$, $i = 1, 2, 3$. If not, by Lemma 4, $|z_1 z_2 z_3| < |Z_1(\zeta)Z_2(\zeta)Z_3(\zeta)| = |\zeta|$, a contradiction. Thus $z = T_{\theta_1,\theta_2} \circ \phi(\zeta) \in \Delta_{\theta_1,\theta_2}$ for some (θ_1, θ_2). In the second case, if $z \notin \Delta_1 \cup \Delta_2$, take a neighborhood U in M of the polynomially convex set $\partial\Delta_1 \cup \partial\Delta_2$ and a polynomial P so that $|P(z)| > 1$ while $|P| < 1$ on U. Then argue as in the proof of Lemma 2, taking $\zeta^0 = 0$, to conclude that $z \notin \widehat{M}$, a contradiction. We are finished with the proof of Theorem 1.

Acknowledgment

I wish to thank W. Rudin and the referee for pointing out several errors in a preliminary version of this paper.

References

1. P. Ahern and W. Rudin, *Totally real imbeddings of S^3 in $\mathbb{C}^3$*, Proc. Amer. Math. Soc. **94** (1985), 460–462.
2. ——, *Hulls of 3-spheres in $\mathbb{C}^3$*, preprint, 1991.
3. H. Alexander, *A note on polynomial hulls*, Proc. Amer. Math. Soc. **33** (1972), 389–391.
4. ——, *Polynomial hulls of graphs*, Pacific J. Math. **147** (1991), 201–212.
5. J. Anderson, *Polynomial hulls of sets invariant under an action of the special unitary group*, Canad. J. Math. **40** (1988), 1256–1271.
6. E. Bishop, *Differentiable manifolds in complex euclidean space*, Duke J. Math. **32** (1965), 1–21.
7. F. Forstneric, *A totally real three-sphere in $\mathbb{C}^3$ bounding a family of analytic disks*, Proc. Amer. Math. Soc. **108** (1990), 887–892.
8. T. Gamelin, *Hartogs series, Hartogs functions, and Jensen measures*, Spaces of Analytic Functions, Kristiansand 1975, Lecture Notes in Math., vol. 512, Springer-Verlag, Berlin and New York, 1976.
9. M. L. Gromov, *Partial differential relations*, Springer-Verlag, Berlin and New York, 1986.
10. J. Wermer, *Subharmonicity and hulls*, Pacific J. Math. **58** (1975), 283–290.

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