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DIMENSIONALLY NILPOTENT JORDAN ALGEBRAS

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ABSTRACT. An algebra A of dimension n is called dimensionally nilpotent if it has a nilpotent derivation ∂ with the property that $\partial^{n-1} \neq 0$. Here we show that a dimensionally nilpotent Jordan algebra A over a perfect field of characteristic not 2 or 3 is either (i) nilpotent, or (ii) one-dimensional modulo its maximal nilpotent ideal. This result is also extended to noncommutative Jordan algebras.

1. INTRODUCTION

The structure of dimensionally nilpotent Lie algebras was studied by Leger and Manley [7] for characteristic 0, and by the author [9–11] for characteristic $p > 5$. Here we investigate dimensionally nilpotent Jordan algebras in §2 and noncommutative Jordan algebras in §3. We assume throughout that F is a field of characteristic not 2 or 3.

For our result, we need some information about the derivations of the Jordan algebra F_q^+, where F_q is the associative algebra of $q \times q$ matrices over F, and F_q^+ is the same algebra under the Jordan product $x \circ y = (xy \times yx)/2$. We recall that the derivations of F_q^+ were found for characteristic 0 by Jacobson [4], and for characteristic p with q not divisible by p by Harris [3]. The case when q is divisible by p, which is what we need, is a special case of Corollary 4.9 of [1] (see also I. N. Herstein, Topics in Ring Theory, Univ. of Chicago Press, Chicago, 1969, p. 55). Specifically, we require

Proposition 1.1. *When $q \geq 3$, the derivations of F_q^+ are exactly the derivations of F_q, namely, the maps of the form ad_X for $x \in F_q$ where $\mathrm{ad}_x y = [x, y] = xy - yx$.*

We also need the following result, which is obvious from the form of ∂.

Lemma 1.2. *If B and C are two ∂-invariant subspaces of a dimensionally nilpotent algebra A, then either $B \subset C$ or $C \subset B$.*

An algebra is called derivation simple if it contains no proper ideals that are invariant under all derivations.

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Proposition 1.3. *Let A be a derivation simple dimensionally nilpotent algebra. Then either A is simple, or* $\operatorname{char} F = p > 0$ *and A is isomorphic to $S \otimes B_r$, where S is a simple dimensionally nilpotent algebra, and B_r is the truncated polynomial algebra on r indeterminates.*

Proof. Although the proof is virtually the same as the proof of Proposition 2.1 of [9], we give it here for completeness. Suppose that A satisfies the hypothesis, i.e., that A has no proper ideals that are invariant under all derivations. It follows from Block [2] that either A is simple, or the characteristic is $p > 0$ and $A \cong S \otimes B_r$, where S is a simple Lie algebra and B_r is the truncated polynomial algebra on r indeterminates. Writing $S' = S \otimes B_{r-1}$, we have $A \cong S' \otimes B_1$. In order to prove Proposition 1.3 by induction on the dimension of A, it is sufficient to establish that A dimensionally nilpotent implies that S' is also. If the field F is chosen so that S is central simple, the derivation algebra of $S' \otimes B_1$ is known [2] to be $\operatorname{Der} S' \otimes B_1 + 1 \otimes \operatorname{Der} B_1$. In the representation of ∂ as an element of $\operatorname{Der} S' \otimes B_1 + 1 \otimes \operatorname{Der} B_1$, we can think of the element of $\operatorname{Der} B_1$ as having the form $g(x)\partial_x$ for some $g(x) \in B_1$ where ∂_x is partial differentiation with respect to x in B_1. We say that a derivation of A is of type 1 if $g(x)$ has a nonzero constant term and of type 2 otherwise.

If ∂ is of type 2, then for $s \in S$, $\partial(s \otimes x^j)$ has no terms of lower degree in x than j. Thus, $S' \otimes x^{p-1}$ is a ∂-invariant subalgebra of A. If ∂ acts nilpotently with a one-dimensional eigenspace on A, then the same is true on $S' \otimes x^{p-1}$; hence S' is dimensionally nilpotent under the induced action of ∂ on S'.

We may suppose that ∂ is of type 1, so that $\partial = a\partial_x + \partial_2$ where $a \in F$ is nonzero and where $\partial_2(S' \otimes x^j) \subset S' \otimes x^j B_1$. For any derivation ∂' of type 1, let $C_{p-1} = S' \otimes x^{p-1}$, and define $C_i = \partial' C_{i+1}$ by downward induction on i. Because of the form of a derivation of type 1, no element of C_{i+1} is annihilated by ∂' for $i \geq 0$. In fact, no nonzero element of $\sum_{i>0} C_i$ is annihilated by ∂'. Clearly A is the vector space direct sum of the C_i's for $0 \leq i \leq p-1$. The form of the elements of each C_i for $i \leq p-2$ depends on the derivation ∂', but the space $\sum_{i \geq s} C_i$ depends only on s and is independent of the particular derivation of type 1 used to define it.

Let ∂_j denote the p^j power of our derivation ∂, and let r be the largest integer such that $p^r < \dim A$. Then $\partial_{r+1} = 0$ is of type 2. Since we are assuming that $\partial_0 = \partial$ is of type 1, there must exist an integer k such that ∂_k is of type 1 and ∂_{k+1} is of type 2. Then, defining the spaces C_i for the derivation ∂_k, we have $\partial_{k+1} C_{p-1} \subset C_{p-1}$; so $\partial_k C_0 = (\partial_k)^p C_{p-1} \subset C_{p-1}$. If $u = u_0 + u_1 + \cdots + u_{p-1} \in \operatorname{Ker} \partial_k$, then $0 = \partial_k u_0 + \partial_k u_1 + \cdots + \partial_k u_{p-1}$, which implies that $0 = u_1 = u_2 = \cdots = u_{p-1}$ and that $u = u_0 \in \operatorname{Ker} \partial_k$. Hence, $\dim \operatorname{Ker} \partial_k = p^k \leq \dim S'$.

Suppose first that $k > 1$, and let $C'_0, \dots, C'_{p-1}$ be the C_i's determined by ∂. Then an element $w \in A$ which is not in ∂A is not in $\sum_{i \neq p-1} C'_i$, and so can be taken to be in $C'_{p-1} = C_{p-1}$. If $n = \dim A$, the span of the elements of the form $\partial^{n-j} w$ for $1 \leq j \leq p^k$ is exactly the roots of ∂_k, and they are all in C_0 by the last paragraph. Hence the span G of the elements of the form $\partial^{n-j} w$ for $p^k + 1 \leq j \leq 2p^k$ is contained in $C_1 \subset \sum_{i>0} C_i = \sum_{i>0} C'_i$. In fact, each of the elements $\partial^l w$ spanning G has a nonzero component in C'_1. But then $\partial^{l+1}w = \partial(\partial^l w) \notin \sum_{i>0} C_i$, which is a contradiction since $\dim G > 1$. This rules out the case $k > 0$.

When $k = 1$, we have $\partial^{n-i}w \in C_{i-1}$ for $1 \le i \le p$, from which it is clear that $\dim C_{p-1} \cap \operatorname{Ker}\partial_1 = 1$. Then the action of ∂_1 on C_{p-1} makes S' into a dimensionally nilpotent Lie algebra using the nilpotent derivation ∂' on S' induced by ∂_1. □

Lemma 1.4. *Let A be a simple power associative algebra of dimension greater than one containing an identity element 1. Then $\operatorname{char} F = p > 0$ and $\dim A = p^r$ for some positive integer r, and A contains an element z with $z^p = 0$ and $z^{p-1} \neq 0$.*

Proof. Since 1 is annihilated by all derivations, it spans the one-dimensional subspace of A annihilated by ∂. Let z_1 be such that $\partial(z_1) = 1$. Then $\partial(z_1^i) = iz_1^{i-1}$, showing that $z_1^i \neq 0$ if $iz_1^{i-1} \neq 0$. If $\operatorname{char} F = 0$, then the powers of z_1 are linearly independent and the dimension of the subalgebra generated by z_1 must be infinite, contrary to the hypothesis. Thus, $\operatorname{char} F = p > 0$, and we see that 1, z_1, $z_1^2, \dots, z_1^{p-1}$ are linearly independent. Since $\partial(z_1^p) = 0$, it follows that z_1^p is a multiple of 1. After subtracting an appropriate multiple of 1 from z_1, we can suppose that $z_1^p = 0$.

Now suppose that $z_1, z_2, \dots, z_k$ are given, and $0 \le i \le p^k - 1$, that $z^{\langle i \rangle} = z_1^{i_1} z_2^{i_2} \cdots z_k^{i_k}$ where the products of powers are associated from the left end, and where $i = i_1 + i_2 p + \cdots + i_k p^{k-1}$ is the expansion of i in the base p. Let B_k' be the span of the $z^{\langle i \rangle}$ with $0 \le i \le p^k - 1$. Assume inductively that $\partial(z^{\langle i \rangle})$ is a nonzero multiple of $z^{\langle i-1 \rangle}$ modulo z's with lower superscripts, and that B_k' is a subalgebra of A. If $B_k' \neq A$, and if $q = p^k$, then there exists an element z_{k+1} such that $\partial(z_{k+1}) = z^{\langle q-1 \rangle}$. Thus, $\partial^q(z_{k+1}) = a1$ for some nonzero $a \in F$, and $\partial^q(z_{k+1}^i) = iaz_{k+1}^{i-1}$, from which we see that $z_{k+1}^i \neq 0$ for $1 \le i \le p-1$ and that $z_{k+1}^p \in B_k'$. Extend the definition of $z^{\langle i \rangle}$ by letting $z^{\langle i \rangle} = z_1^{i_1} z_2^{i_2} \cdots z_{k+1}^{i_{k+1}}$ for $0 \le i \le p^{k+1} - 1$, where $i = i_1 + i_2 p + \cdots + i_{k+1} p^k$ is the expansion of i in the base p. Let B_{k+1}' be the span of the $z^{\langle i \rangle}$ for $0 \le i \le p^{k+1} - 1$.

To complete the inductive step, we must show for $0 \le i \le p^{k+1} - 1$ that $\partial(z^{\langle i \rangle})$ is a nonzero multiple of $z^{\langle i-1 \rangle}$ modulo lower terms, and that B_{k+1}' is a subalgebra. We establish the first by induction on i, noting that it is assumed to be true when $i < p^k$:

$$\begin{aligned}\partial^q \partial(z^{\langle i \rangle}) &= \partial[\partial^q(z_1^{i_1} z_2^{i_2} \cdots z_k^{i_k}) z_{k+1}^{i_{k+1}} + (z_1^{i_1} z_2^{i_2} \cdots z_k^{i_k}) \partial^q(z_{k+1}^{i_{k+1}})] \\ &= \partial[(z_1^{i_1} z_2^{i_2} \cdots z_k^{i_k}) i_{k+1} a z_{k+1}^{i_{k+1}-1}] = i_{k+1} a \partial(z^{\langle i-q \rangle}).\end{aligned}$$

Then B_{k+1}' can be characterized by the fact that it is the set of elements of A that are annihilated by ∂^{qp}, from which it follows that B_{k+1}' is a subalgebra.

We have shown by induction that each B_k' is a subalgebra, and that if $B_k' \neq A$ then B_{k+1}' exists inside A. Thus, $A = B_k'$ for some k. Clearly, $\dim B_k' = p^k$, so that $\dim A = p^k$. □

2. The Jordan case

We can now establish

Theorem 2.1. *If A is a dimensionally nilpotent Jordan algebra over a perfect field F with* $\operatorname{char} F \neq 2, 3$, *then either A is nilpotent or* $\dim(A/\operatorname{Rad} A) = 1$.

Proof. If E is the algebraic closure of F, then $A_E = A \otimes_F E$ is also dimensionally nilpotent. Further, if A_E satisfies the conclusion of Theorem 2.1, then so does A. Thus it is sufficient to establish Theorem 2.1 in the case when F is algebraically closed. We may also suppose that the maximal ∂-invariant nilpotent ideal N of A has been divided out, since A satisfies the conclusion of the theorem if A/N does. Then A is a direct sum of ∂-invariant algebras by Block [2] (or Ravisankar [13]). It follows from Lemma 1.2 that A is ∂-simple. In view of Proposition 1.3, we need only consider the case when A is simple.

Thus it remains to show that if A is simple then $\dim A = 1$. Since simple Jordan algebras have identity elements, the hypotheses of Proposition 1.4 are satisfied. Then, if $\dim A > 1$, $\operatorname{char} F = p > 0$ and A contains an element z that is nilpotent of order exactly p. Now, A can have no such element if A is a simple algebra determined by a symmetric bilinear form, or if A is exceptional simple. If $A \cong H(F_q^+, \tau)$ for some involution τ of F_q^+, then $\dim A$ is either $q(q-1)/2$ or $q(q+1)/2$, depending on the type of τ. But $\dim A$ is a power of p by Proposition 4.1, and neither of these numbers can be a prime power. It follows that A has the form F_q^+, where q is a power of p. Then the nilpotent derivation ∂ on A with the property that $\partial^{n-1} \neq 0$ must have the form $\partial = \operatorname{ad}_x$ for some $x \in A$ by Proposition 1.1. Noting that $\partial(x) = 0 = \partial(1)$, and that the subspace annihilated by ∂ is one-dimensional since $\partial^{n-1} \neq 0$, we see that x is a multiple of 1. But then $\partial(A) = 0$, implying that $n = 1$. □

3. The noncommutative Jordan case

Here we establish

Theorem 3.1. *Let A be a dimensionally nilpotent noncommutative Jordan algebra over a perfect field of characteristic not* 2 *or* 3. *Then, modulo its maximal nil ideal, A is either zero-dimensional, one-dimensional, or a nodal algebra.*

Proof. Let ∂ be a dimensionally nilpotent derivation of A. If A has a ∂-invariant nil ideal N, and if A/N satisfies the conclusion of the theorem, then so does A. Thus we may suppose that A is ∂-semisimple. By Block [2], A is a direct sum of ∂-simple algebras, so that A is ∂-simple by Lemma 1.2. It follows from Proposition 1.3 that either A is simple or $\operatorname{char} F = p$ and $A \cong B_r$. Since the latter case satisfies the conclusion of Theorem 3.1, we may suppose that A is simple. If A (after possibly making a scalar extension) has two orthogonal idempotents, then so does the Jordan algebra A^+. But ∂ is also a dimensionally nilpotent derivation of A^+, and so A^+ cannot have two orthogonal idempotents by Theorem 2.1. It follows [14, p. 143] that A must be either one-dimensional or a nodal algebra. □

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