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ON THE ALMOST SPLIT SEQUENCES FOR RELATIVELY PROJECTIVE MODULES OVER A FINITE GROUP

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ABSTRACT. Let G be a finite group with a subgroup H. Given a field k of characteristic p dividing the order of G, denote by $\operatorname{mod} kG$ the category of finite-dimensional over k left G-modules, and let $\mathscr{C}$ be the full subcategory of $\operatorname{mod} kG$ determined by the relatively projective modules. Let $0 \to L \to M \to N \to 0$ be an exact sequence in $\operatorname{mod} kG$ with $L, M, N \in \mathscr{C}$. It is proved that the sequence is an almost split sequence in $\mathscr{C}$ if and only if it is an almost split sequence in $\operatorname{mod} kG$. This implies, together with a recent result of Carlson and Happel, that $\mathscr{C}$ has almost split sequences if and only if it is closed under extensions, i.e., if and only if p is coprime to either the order of H or the index of H in G.

Throughout the paper, we fix an artin algebra Λ [1] with Jacobson radical $\mathbf{r}$ and denote by $\operatorname{mod} \Lambda$ the category of finitely generated left Λ-modules. We also fix $\mathscr{C}$, a full subcategory of $\operatorname{mod} \Lambda$ closed under isomorphisms, direct sums, and direct summands. Denote by $\operatorname{ind}\mathscr{C}$ the set of pairwise nonisomorphic indecomposable modules in $\mathscr{C}$, and write $\operatorname{ind} \Lambda$ for $\operatorname{ind}(\operatorname{mod} \Lambda)$. We say that $\mathscr{C}$ is closed under DTr and TrD if for each $C \in \operatorname{ind}\mathscr{C}$, $DTrC$ and $TrDC$ [2] are in $\mathscr{C}$. We freely use the notions of a functorially finite subcategory and of a left or right almost split morphism in $\mathscr{C}$, introduced in [5, 6] as a generalization of the corresponding notions in [2].

The only modification is that we replace Ext-projective and Ext-injective modules in $\mathscr{C}$ [5] by extension projective and extension injective modules, respectively, which are defined as follows. A module $N \in \mathscr{C}$ is called *extension projective* in $\mathscr{C}$ if every exact sequence $0 \to X \to Y \to N \to 0$ in $\mathscr{C}$ splits. Recall that a module $N \in \mathscr{C}$ is called Ext-projective in $\mathscr{C}$ if $\operatorname{Ext}^1_\Lambda(N, X) = 0$ for all $X \in \mathscr{C}$. Clearly, every Ext-projective module is extension projective. On the other hand, in the context of [5], $\mathscr{C}$ is closed under extensions, i.e., in every exact sequence $0 \to X \to Y \to Z \to 0$ with $X, Z \in \mathscr{C}$, we have $Y \in \mathscr{C}$ so that a module is Ext-projective if and only if it is extension projective. Extension injective modules in $\mathscr{C}$ are introduced dually.

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Recall that an exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ in $\mathscr{C}$ is called *almost split* in $\mathscr{C}$ if f is a left almost split morphism in $\mathscr{C}$ and g is a right almost split morphism in $\mathscr{C}$. $\mathscr{C}$ is said to have almost split sequences if it satisfies the following conditions:

(a) $\mathscr{C}$ has almost split morphisms.
(b) If $N \in \operatorname{ind} \mathscr{C}$ is not extension projective, then there is an almost split sequence $0 \to L \to M \to N \to 0$ in $\mathscr{C}$.
(c) Dual to (b).

It is obvious that if an exact sequence $0 \to L \to M \to N \to 0$ in $\mathscr{C}$ is almost split in $\operatorname{mod} \Lambda$, then it is almost split in $\mathscr{C}$. The converse is true only under certain conditions, as was pointed out by K. W. Roggenkamp, whose short proof replaces here an argument of the author.

Propositon 1. (a) *Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be an almost split sequence in $\mathscr{C}$, and let $0 \to U \xrightarrow{s} V \xrightarrow{t} N \to 0$ be the almost split sequence in* $\operatorname{mod} \Lambda$ *with right end N, where $U \in \mathscr{C}$. Then the sequences are isomorphic.*

(b) *If $\mathscr{C}$ is closed under DTr and TrD, then a short exact sequence in $\mathscr{C}$ is almost split in $\mathscr{C}$ if and only if it is almost split in* $\operatorname{mod} \Lambda$.

Proof. (a) Since g is not a splittable epimorphism, we have the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "h"'] & M \arrow[r, "g"] \arrow[d, "j"'] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U \arrow[r, "s"] & V \arrow[r, "t"] & N \arrow[r] & 0
\end{tikzcd}$$

If h is not a splittable monomorphism, it can be extended to M because $U \in \mathscr{C}$ and f is a left almost split morphism in $\mathscr{C}$. Then the bottom row splits, a contradiction. Thus, h is a splittable monomorphism and must be an isomorphism because $U \in \operatorname{ind} \Lambda$.

(b) Follows from (a) and its dual. □

Proposition 2. *Let Λ be connected. Suppose that*

(i) *$\mathscr{C}$ has almost split sequences and is closed under DTr and TrD,*
(ii) *all extension projectives in $\mathscr{C}$ are projective in* $\operatorname{mod} \Lambda$,
(iii) *all extension injectives in $\mathscr{C}$ are injective in* $\operatorname{mod} \Lambda$,
(iv) *$\mathscr{C}$ is functorially finite.*

Then either $\mathscr{C} =$ $\operatorname{mod} \Lambda$ *or $\mathscr{C}$ consists of projective-injective modules in* $\operatorname{mod} \Lambda$; *in particular, $\mathscr{C}$ is closed under extensions.*

Proof. Let $\mathscr{C}$ satisfy (i) and (ii), and suppose $L \in \operatorname{ind} \mathscr{C}$ is not extension injective. Show that if $X \to L$ and $L \to Y$ are irreducible maps with $X, Y \in \operatorname{ind} \Lambda$, then $X, Y \in \mathscr{C}$. Really, by assumption, there is an almost split sequence $0 \to L \to M \to N \to 0$ in $\mathscr{C}$ that is almost split in $\operatorname{mod} \Lambda$ by Proposition 1(b). Then $Y \in \mathscr{C}$ for Y is a direct summand of M. If L is not extension projective in $\mathscr{C}$, there is an almost split sequence $0 \to U \to V \to L \to 0$ in $\mathscr{C}$ and $X \in \mathscr{C}$ for X is a direct summand of V. If L is extension projective, it is projective and X is not injective in $\operatorname{mod} \Lambda$. Then there exists an almost split sequence $0 \to X \to L \oplus L' \to Z \to 0$ in mod Λ with Z a direct summand of M, whence $Z \in \mathscr{C}$ and $X \in \mathscr{C}$ by (i).

From the preceding argument and its dual, we conclude that if $L \in \operatorname{ind}\mathscr{C}$ is not projective-injective in mod Λ, then for all irreducible maps $X \to L$ and $L \to Y$, we have $X, Y \in \mathscr{C}$. Let $\mathscr{D} = \{P_i | i = 1, \dots, s\}$ be the set of all those pairwise nonisomorphic projective-injective modules in $\operatorname{ind}\Lambda$ for which the ends of the almost split sequence $0 \to \mathbf{r}P_i \to \mathbf{r}P_i/\operatorname{soc} P_i \oplus P_i \to P_i/\operatorname{soc} P_i \to 0$ [3, Proposition 4.11] are not in $\mathscr{C}$. Then $\mathscr{C}_{\mathscr{D}}$, the full subcategory of $\mathscr{C}$ determined by all modules having no direct summands in $\mathscr{D}$, has the following property. If $X \to Y$ is an irreducible map with $X, Y \in \operatorname{ind}\Lambda$, then $X \in \mathscr{C}_{\mathscr{D}}$ if and only if $Y \in \mathscr{C}_{\mathscr{D}}$.

Now suppose $\mathscr{C} \neq$ mod Λ is functorially finite. Then $\mathscr{C}_{\mathscr{D}}$ is functorially finite by [6, Propositon 3.13], and, according to [8, Corollary 2.2] and its dual, we have $\operatorname{Hom}_\Lambda(C, W) = 0 = \operatorname{Hom}_\Lambda(W, C)$ for all $C \in \operatorname{ind}\mathscr{C}_{\mathscr{D}}$ and $W \in (\operatorname{ind}\Lambda) - (\operatorname{ind}\mathscr{C}_{\mathscr{D}})$. Since Λ is connected, indecomposable projective Λ-modules must either all be in $\operatorname{ind}\mathscr{C}_{\mathscr{D}}$ or all belong to $(\operatorname{ind}\Lambda) - (\operatorname{ind}\mathscr{C}_{\mathscr{D}}) \neq \varnothing$. Clearly, the latter holds, i.e., $\operatorname{ind}\mathscr{C} = \mathscr{D}$. □

Remark 3. Proposition 2 is false without assumptions (ii) and (iii). Really, if Λ is the group algebra of a finite group of order 2 over a field of characteristic 2 and $\mathscr{C}$ is the additive subcategory of mod Λ generated by the trivial module, then $\mathscr{C}$ satisifies (i) and (iv) but is not closed under extensions.

Let G be a finite group of order $|G|$ with a subgroup H of index $[G : H]$. Let k be a field of characteristic p dividing $|G|$. From now on, put $\Lambda = kG, \Gamma = kH$, and denote by $\mathscr{C}$ the full subcategory of relatively projective modules [9, 10] in mod Λ, i.e., of all modules isomorphic to a direct summand of $\Lambda \otimes_\Gamma X$ for some $X \in$ mod Γ. It is well known that $\mathscr{C}$ is functorially finite, and the extension projective modules, as well as the extension injective modules in $\mathscr{C}$, are projective in mod Λ. In addition, $\mathscr{C}$ is closed under DTr and TrD by [4, p. 550]. Thus, Proposition 1 and 2 hold for every block of Λ. Since the necessary and sufficient conditions for the closure of $\mathscr{C}$ under extensions are well known, we can restate Proposition 2 as follows.

Corollary 4. *$\mathscr{C}$ has almost split sequences if and only if p is coprime to either $|H|$ or $[G : H]$.*

Now suppose that $\mathscr{C}$ does not have almost split sequences. Since $\mathscr{C}$ is self-dual, it follows from Proposition 1 and from [7, Theorem 1.2], where the term extension projective should have been used instead of Ext-projective, that

(i) for some nonprojective $N \in \operatorname{ind}\mathscr{C}$, the middle term of the almost split sequence $0 \to L \to M \to N \to 0$ in mod Λ is not in $\mathscr{C}$.

(ii) For some nonprojective $N' \in \operatorname{ind}\mathscr{C}$, the kernel of one (and every!) right almost split morphism $V \to N' \to 0$ in $\mathscr{C}$ is not in $\mathscr{C}$. Our next statement shows that for a given nonprojective module in $\operatorname{ind}\mathscr{C}$, (i) and (ii) are equivalent.

Proposition 5. *Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be an exact sequence in* mod Λ *with $L \in \mathscr{C}$. Then*

(a) *In the exact sequence $0 \to K \to \Lambda \otimes_\Gamma M \xrightarrow{gm} N \to 0$, we have $K \simeq L \oplus X(M)$, where $m : \Lambda \otimes_\Gamma M \to M$ is the multiplication map and $X(M) = \operatorname{Ker} m$.*

(b) *$M \in \mathscr{C}$ if and only $K \in \mathscr{C}$.*

(c) *If $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is an almost split sequence in $\operatorname{mod}\Lambda$, then gm is a right almost split morphism in $\mathscr{C}$.*

Proof. (a) We have the following exact commutative diagram of Λ-modules:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & X(M) \arrow[d] \arrow[r, phantom, "="] & X(M) \arrow[d] & & \\
0 \arrow[r] & K \arrow[r, "h"] \arrow[d, "j"'] & \Lambda \otimes_\Gamma M \arrow[r, "gm"] \arrow[d, "m"'] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] \arrow[d] & N \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

The middle column splits as a sequence of Γ-modules. Since L is relatively projective, f can be lifted to $\Lambda \otimes_\Gamma M$, and the left column splits.

(b) By the Krull-Schmidt theorem, it follows from (a) that $K \in \mathscr{C}$ if and only if $X(M) \in \mathscr{C}$, because $L \in \mathscr{C}$. Since every module in $\mathscr{C}$ is relatively injective and the middle column of the diagram splits over Γ, $X(M) \in \mathscr{C}$ if and only if m is a splittable epimorphism over Λ, i.e., if and only if $M \in \mathscr{C}$.

(c) Clearly, $N \in \mathscr{C}$, and gm is a right almost split morphism in $\mathscr{C}$ by the proof of [6, Proposition 3.10]. □

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