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BUZANO'S INEQUALITY AND BOUNDS FOR ROOTS OF ALGEBRAIC EQUATIONS

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Dedicated to Professor Tsuyoshi Ando on his 60th birthday

ABSTRACT. A new bound for roots of algebraic equations will be given as a consequence of an inequality due to Buzano.

1. INTRODUCTION

Buzano [1] obtained an extension of Schwarz's inequality: *If* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{x}$ *are vectors in an inner product space* $\mathscr{H}$, *then*

$$|\langle \mathbf{a} \mid \mathbf{x}\rangle \cdot \langle \mathbf{x} \mid \mathbf{b}\rangle| \leq \frac{\|\mathbf{a}\| \cdot \|\mathbf{b}\| + |\langle \mathbf{a} \mid \mathbf{b}\rangle|}{2} \|\mathbf{x}\|^2 . \tag{1}$$

Since her proof is a little complicated, a new, simple proof will be given with the equality condition.

Let P be an orthogonal projection on a subspace of an inner product space $\mathscr{H}$. If $\mathbf{a}, \mathbf{b} \in \mathscr{H}$, then the usual Schwarz's inequality implies that

$$|\langle (2P - I)\mathbf{a} \mid \mathbf{b}\rangle| \leq \|\mathbf{a}\| \cdot \|\mathbf{b}\| . \tag{2}$$

Let $(\mathbf{u} \otimes \mathbf{v})\mathbf{w} = \langle \mathbf{w} \mid \mathbf{v}\rangle \mathbf{u}$ $(\mathbf{w} \in \mathscr{H})$. Then the operator $\mathbf{x} \otimes \mathbf{x}$ is an orthogonal projection if $\|\mathbf{x}\| = 1$, and hence $|\langle (2\mathbf{x} \otimes \mathbf{x} - I)\mathbf{a} \mid \mathbf{b}\rangle| \leq \|\mathbf{a}\| \cdot \|\mathbf{b}\|$, which implies the required one:

$$2|\langle (\mathbf{x} \otimes \mathbf{x})\mathbf{a} \mid \mathbf{b}\rangle| - |\langle \mathbf{a} \mid \mathbf{b}\rangle| \leq |\langle (2\mathbf{x} \otimes \mathbf{x} - I)\mathbf{a} \mid \mathbf{b}\rangle| \leq \|\mathbf{a}\| \cdot \|\mathbf{b}\| .$$

The equality holds iff two inequality signs in the last line turn out to be equal, from which one obtains the equality condition: The equality in (1) holds if

$$\mathbf{x} = \begin{cases} \alpha \left(\dfrac{\mathbf{a}}{\|\mathbf{a}\|} + \dfrac{\langle \mathbf{a} \mid \mathbf{b}\rangle}{|\langle \mathbf{a} \mid \mathbf{b}\rangle|} \dfrac{\mathbf{b}}{\|\mathbf{b}\|} \right), & \text{when } \langle \mathbf{a} \mid \mathbf{b}\rangle \neq 0, \\ \alpha \left(\dfrac{\mathbf{a}}{\|\mathbf{a}\|} + \beta \dfrac{\mathbf{b}}{\|\mathbf{b}\|} \right), & \text{when } \langle \mathbf{a} \mid \mathbf{b}\rangle = 0, \end{cases}$$

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where α, β are complex numbers with $|\beta| = 1$.

Define the numerical radius $w(T)$ of an operator T acting on $\mathscr{H}$ by

$$w(T) = \sup\{|\langle T\mathbf{x} \mid \mathbf{x}\rangle| : \|\mathbf{x}\| = 1\}.$$

Thus Buzano's inequality with the equality condition implies at once the following theorem.

Theorem 1. *If $T = \mathbf{a} \otimes \mathbf{b}$ is a linear operator of rank one, then*

$$w(T) = \frac{\|\mathbf{a}\| \cdot \|\mathbf{b}\| + |\langle \mathbf{a} \mid \mathbf{b}\rangle|}{2}.$$

In this paper, Theorem 1 will be applied to obtain a bound for roots of algebraic equations. Other comments on Buzano's inequality will be published elsewhere.

2. Bounds for roots of algebraic equations

Let

$$C = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{pmatrix}$$

be the companion matrix associated with the algebraic equation

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0 = 0. \tag{3}$$

It is well known (cf. [5]) that the set of roots of (3) is identical with the spectrum $\sigma(C)$ of C. In [3], it was shown that those classical bounds for roots were obtained as operator norms of the companion matrix C (cf. [4]). Since the numerical range $W(T) = \{\langle T\mathbf{x} \mid \mathbf{x}\rangle : \|\mathbf{x}\| = 1\}$ contains $\sigma(T)$, it is expected that an estimation of $w(C)$ gives a new bound for roots of (3).

Theorem 2. *If z is a root of an algebraic equation* (3) *then*

$$|z| \le \cos\frac{\pi}{n+1} + \frac{\sqrt{\sum_{i=0}^{n-1} |a_i|^2} + |a_{n-1}|}{2}. \tag{4}$$

Proof. Since $C = S - \mathbf{e}_1 \otimes \mathbf{a}$, where

$$\mathbf{a} = \begin{pmatrix} \overline{a_{n-1}} \\ \overline{a_{n-2}} \\ \vdots \\ \overline{a_0} \end{pmatrix}, \quad \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \text{and} \quad S = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & \cdots & & \cdots & & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

one has only to estimate the value

$$w(C) = w(S - \mathbf{e}_1 \otimes \bar{\mathbf{a}}) \le w(S) + w(\mathbf{e}_1 \otimes \bar{\mathbf{a}}) = w(S) + \frac{\|\mathbf{a}\| + |a_{n-1}|}{2}.$$

To estimate $w(S)$, one can consult with the recent paper of Davidson and Holbrook [2].

Finally a comparison with the bound due to Carmichael-Mason (cf. [5]) will be given: *If z is a root of* (3) *then* $|z| \le B_{CM} = \sqrt{1 + \sum_{i=0}^{n-1} |a_i|^2}$. Their bound is not always better than the one in Theorem 2, and vice versa. It is obvious that if the second leading coefficient vanishes and $\|\mathbf{a}\|$ is fairly large, then the new bound is better than B_{CM}.

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