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TWO WEIGHT NORM INEQUALITIES FOR FRACTIONAL ONE-SIDED MAXIMAL OPERATORS

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ABSTRACT. In this paper we introduce a new maximal function, the dyadic one-sided maximal function. We prove that this maximal function is equivalent to the one-sided maximal function studied by the authors and Ortega in *Weighted inequalities for one-sided maximal functions* (Trans. Amer. Math. Soc. **319** (1990)) and by Sawyer in *Weighted inequalities for the one-sided Hardy-Littlewood maximal functions* (Trans. Amer. Math. Soc. **297** (1986)), but our function, being dyadic, is much easier to deal with, and it allows us to study fractional maximal operators. In this way we obtain a geometric proof of the characterization of the good weights for fractional maximal operators, answering a question raised by Andersen and Sawyer in *Weighted norm inequalities for the Riemann-Liouville and Weyl fractional integral operators* (Trans. Amer. Math. Soc. **308** (1988)). Our methods, avoiding complex interpolation, give also the case of different weights for the fractional maximal operator, which was an open problem.

1. INTRODUCTION

In [1] Andersen and Sawyer characterized the good weights for the fractional maximal function M_α^+ using complex interpolation and, as a consequence, were able to characterize the good weights for the fractional integral operators. Their methods seem to be restricted to the case of equal weights and raise the question of obtaining a geometric proof of the characterization of the good weights for M_α^+.

We introduce a dyadic one-sided maximal function $M_{\alpha,D}^+$, and prove that it is pointwise equivalent to M_α^+; furthermore, since our maximal function is dyadic, Sawyer's original technique [3] can be used to characterize the pairs of weights for which it is bounded (even in the case of different weights). We obtain a general condition and prove that in the case of equal weights it is equivalent to condition (1.5) in [1]. In this way we give a new proof of Theorem 1 in [1].

Throughout this paper C will denote a positive constant, not necessarily the same at each occurrence. If $p > 1$, its conjugate exponent will be denoted by p'. For any measurable set A and any positive function g, χ_A will denote the

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characteristic function of A, $|A|$ its Lebesgue measure, and $g(A)$ the integral of g over A. If $I = [a, b)$ is an interval, then we will denote by I^* the interval $[b, 2b-a)$.

2. The fractional, one-sided, dyadic maximal function

For each x in $\mathbb{R}$, we consider the family of intervals $A_x = \{I = [a, b); I$ is dyadic and $0 \le a - x < b - a\}$. Now for each locally integrable f and $1 > \alpha > 0$, we define the one-sided, dyadic, fractional maximal function $M^+_{\alpha,D}$ as

$$M^+_{\alpha,D}f(x) = \sup\left\{|I|^{\alpha-1}\int_I |f|;\ I \in A_x\right\}. \tag{2.1}$$

The interest of this maximal function lies in the fact that it is equivalent to the usual one-sided fractional maximal function

$$M^+_\alpha f(x) = \sup_{\alpha>0} a^{\alpha-1}\int_x^{x+a} |f|. \tag{2.2}$$

In order to be able to take averages away from x it is convenient to introduce a new maximal function that is equivalent to M^+_α,

$$N^+_\alpha f(x) = \sup_{\alpha>0}\left(\frac{a}{2}\right)^{\alpha-1}\int_{x+a/2}^{x+a} |f|. \tag{2.3}$$

(2.4) **Proposition.**

$$M^+_\alpha f(x) \le (2^{1-\alpha}-1)^{-1}N^+_\alpha f(x),$$
$$N^+_\alpha f(x) \le 2^{1-\alpha}M^+_\alpha f(x).$$

Proof. It is enough to consider positive and bounded f and take sups in the following obvious inequalities:

$$(a/2)^{\alpha-1}\int_{x+a/2}^{x+a} f \le (a/2)^{\alpha-1}\int_x^{x+a} f \le 2^{1-\alpha}M^+_\alpha f(x),$$

$$a^{\alpha-1}\int_x^{x+a} f = a^{\alpha-1}\int_x^{x+a/2} f + a^{\alpha-1}\int_{x+a/2}^{x+a} f \le 2^{\alpha-1}M^+_\alpha f(x) + 2^{\alpha-1}N^+_\alpha f(x).$$

(2.5) **Proposition.** *For each* α *there are two constants* P_α *and* Q_α *such that*

$$M^+_\alpha f(x) \le P_\alpha M^+_{\alpha,D}f(x), \qquad M^+_{\alpha,D}f(x) \le Q_\alpha M^+_\alpha f(x).$$

Proof. Let us fix $f \ge 0$ and x in $\mathbb{R}$. Let $I = [a, b) \in A_x$. Then

$$|I|^{\alpha-1}\int_I f \le (b-x)^{1-\alpha}|I|^{\alpha-1}(b-x)^{\alpha-1}\int_x^b f \le 2^{1-\alpha}M^+_\alpha f(x).$$

Conversely, it is enough to consider the case in which a is of the form 2^k; let I and I^* be two dyadic intervals of length 2^{k-1}, whose union covers $[x + a/2, x + a)$. If $I \cup I^*$ is dyadic then $I \cup I^* \in A_x$ and

$$\int_{x+a/2}^{x+a} f \le \int_{I\cup I^*} f \le 2^{k(1-\alpha)}M^+_{\alpha,D}f(x).$$

If $I \cup I^*$ is not dyadic then let I_1 be the dyadic interval of length 2^k that contains I^*. It is clear that I and I_1 belong to A_x, and then

$$\int_{x+a/2}^{x+a} f \leq \int_I f + \int_{I_1} f \leq (2^{(k-1)(1-\alpha)} + 2^{k(1-\alpha)}) M_{\alpha,D}^+ f(x).$$

In any case,

$$\int_{x+a/2}^{x+a} f \leq 2^{k(1-\alpha)}(1+2^{\alpha-1}) M_{\alpha,D}^+ f(x);$$

therefore, $N_\alpha^+ f(x) \leq 2^{1-\alpha}(1+2^{\alpha-1}) M_{\alpha,D}^+ f(x)$, and the result follows from (2.4).

(2.6) **Theorem.** *For nonnegative functions u, v and $1 < p \leq q$, the following two conditions are equivalent.*

(A) *There exists C such that for every nonnegative f*

$$\left(\int (M_{\alpha,D}^+ f)^q u\right)^{1/q} \leq C \left(\int f^p v\right)^{1/p}.$$

($S_{p,q,\alpha,D}^+$) *There exists C such that for every dyadic interval $I = [a, b)$ such that $\int_{(-\infty,a)} u > 0$, one has*

$$\int_{I\cup I^*} \sigma < \infty \quad \textit{and} \quad \left(\int_{I\cup I^*} (M_{\alpha,D}^+ \sigma\chi_{I^*})^q u\right)^{1/q} \leq C \left(\int_{I^*} \sigma\right)^{1/p} \tag{2.7}$$

where $\sigma = v^{1-p'}$.

Proof. To prove the first part of (2.7) we observe that if for some dyadic interval $I = [a, b)$ is $\int_{I\cup I^*} \sigma = \infty$, while $\int_{(-\infty,a)} u > 0$, then there is a function f in $L_p(v)$ such that $\int_{I\cup I^*} f = \infty$. This implies that for points $x < a$, close to a, $M_{\alpha,D}^+ f(x) = \infty$, contradicting ($A$). For the second part just take $f = \sigma\chi_{I^*}$. The converse is a modification of Sawyer's argument [3]. Without loss of generality, we may assume that the length of our intervals is uniformly bounded. For each integer k, we consider the set $O_k = \{x;\, M_{\alpha,D}^+ f(x) > 2^k\}$. Then for each x in O_k there is a dyadic interval $I_{x,k}$ such that $x \in I_{x,k}$ and $\int_{I_{x,k}^*} f > 2^k |I_{x,k}^*|^{1-\alpha}$. From the definition of $M_{\alpha,D}^+$, it follows that $I_{x,k}$ is contained in O_k. Since our intervals are dyadic, we may choose for each k, a maximal, disjoint collection $I_{j,k}$ such that

(a) $\bigcup I_{j,k} = O_k$,

(b) $\int_{I_{j,k}^*} f > 2^k |I_{j,k}^*|^{1-\alpha}$.

Now if we define $E_{j,k} = \{x \in I_{j,k};\, M_{\alpha,D}^+ f(x) \leq 2^{k+1}\}$, we may write

$$\begin{aligned}\int (M_{\alpha,D}^+ f)^q u &\leq 2^q \sum_{j,k} 2^{kq} u(E_{j,k}) \\ &\leq 2^q \sum_{j,k} u(E_{j,k}) \left(|I_{j,k}^*|^{\alpha-1} \int_{I_{j,k}^*} f\right)^q \\ &= 2^q \sum_{j,k} \gamma_{j,k} \left(\sigma(I_{j,k}^*)^{-1} \int_{I_{j,k}^*} f\sigma^{-1}\sigma\right)^q,\end{aligned}$$

where

$$\gamma_{j,k} = u(E_{j,k}) \left(|I^*_{j,k}|^{\alpha-1} \int_{I^*_{j,k}} \sigma \right)^q .$$

Following the argument in [3], it is enough to prove that the operator T, defined by $Tg = \sigma(I^*_{j,k})^{-1} \int_{I^*_{j,k}} |g|\sigma$ is of weak type $(1, q/p)$ with respect to the measures $\gamma_{j,k}$ in $\mathbb{Z} \times \mathbb{Z}$, and $\sigma\, dx$ in $\mathbb{R}$. We need then to prove that

$$\sum \{\gamma_{j,k} ; Tg(j, k) > \lambda\} \le C \left(\lambda^{-1} \int |g|\sigma \right)^{q/p} .$$

Since our intervals $I^*_{j,k}$ are dyadic, we may choose a maximal collection I^*_i relative to the property $Tg(j, k) > \lambda$. It is clear that for each x in $E_{j,k}$,

$$|I^*_{j,k}|^{\alpha-1} \int_{I^*_{j,k}} \sigma \le M^+_{\alpha,D}(\sigma\chi_{I^*_{j,k}})(x) .$$

Therefore using (2.7) and the fact that $q/p > 1$, we have

$$\begin{aligned} \sum \{\gamma_{j,k} ; Tg(j, k) > \lambda\} &\le \sum_{j,k} \int_{E_{j,k}} u(M^+_{\alpha,D}(\sigma\chi_{I^*_{j,k}})(x))^q \\ &\le \sum_i \sum_{I^*_{j,k} \subseteq I^*_i} \int_{Ej,k} u(M^+_{\alpha,D}(\sigma\chi_{I^*_i})(x))^q ; \end{aligned}$$

but $I^*_{j,k} \subset I^*_i$ implies $E_{j,k} \subset I_i \cup I^*_i$, and then

$$\begin{aligned} \sum \{\gamma_{j,k} ; Tg(j, k) > \lambda\} &\le \sum_i \int_{I_i \cup I^*_i} u(M^+_{\alpha,D}(\sigma\chi_{I^*_i})(x))^q \\ \le C \sum_i \left(\int_{I^*_i} \sigma \right)^{q/p} &\le C \left(\sum_i \int_{I^*_i} \sigma \right)^{q/p} \le C \left(\lambda^{-1} \int |g|\sigma \right)^{q/p} . \end{aligned}$$

Remark. It follows from the proof that although (2.7) is stated only for dyadic intervals, it is equivalent to the same condition for any interval.

$(S^+_{p,q,\alpha,D})$ seems stronger than the usual (S^+_p) condition, but actually they are equivalent.

Proposition. *The following two conditions are equivalent.*

$(S^+_{p,q,\alpha,D})$ *There exists C such that for every interval I, with $\sigma(I \cup I^*)$ finite,*

$$\left(\int_{I \cup I^*} (M^+_{\alpha,D}\sigma\chi_{I^*})^q u \right)^{1/q} \le C \left(\int_{I^*} \sigma \right)^{1/p} .$$

$(S^+_{p,q,\alpha})$ *There exists C such that for every interval I with $\sigma(I)$ finite $(\int_I (M^+_\alpha \sigma\chi_I)^q u)^{1/q} \le C(\int_I \sigma)^{1/p}$.*

Proof. $(S^+_{p,q,\alpha,D})$ implies $(S^+_{p,q,\alpha})$ follows immediately from the equivalence of M^+_α and $M^+_{\alpha,D}$. Conversely, let I be any interval. Then

$$\begin{aligned} \left(\int_{I \cup I^*} (M^+_{\alpha,D}\sigma\chi_{I^*})^q u \right)^{1/q} &\le C \left(\int_I (M^+_\alpha \sigma\chi_{I^*})^q u \right)^{1/q} + C \left(\int_{I^*} (M^+_\alpha \sigma\chi_{I^*})^q u \right)^{1/q} \\ &\le C \left(\int_I (M^+_\alpha \sigma\chi_{I^*})^q u \right)^{1/q} + C \left(\int_{I^*} \sigma \right)^{1/p} . \end{aligned}$$

Therefore, it is enough to show that for every I there is a C such that

$$\left(\int_I (M_\alpha^+\sigma\chi_{I^*})^q u\right)^{1/q} \le C\left(\int_{I^*}\sigma\right)^{1/p}.$$

Let $I=[a,b)$; $I^*=[b,c)$. If $\int_I\sigma \le \int_{I^*}\sigma$ then

$$\begin{aligned}\left(\int_I (M_\alpha^+\sigma\chi_{I^*})^q u\right)^{1/q} &\le \left(\int_{I\cup I^*} (M_\alpha^+\sigma\chi_{I^*})^q u\right)^{1/q}\\ &\le C\left(\int_{I\cup I^*}\sigma\right)^{1/p} \le 2^{1/p}C\left(\int_{I^*}\sigma\right)^{1/p}.\end{aligned}$$

If $\int_I\sigma \ge \int_{I^*}\sigma$, then we choose a sequence $x_0=b>x_1>x_2>\dots>x_k>$ $,\dots, x_{N-1}>x_N=a$ such that for $k=0,1,\dots,N-1$, $\int_{x_k}^c\sigma = 2^k\int_b^c\sigma$ and $\int_a^c\sigma = r\int_b^c\sigma$, $2^{N-1}<r\le 2^N$. It follows that $\int_{x_k}^{x_{k-1}}\sigma = 2^{k-1}\int_b^c\sigma$, $0<k<N$, while $\int_a^{x_{N-1}}\sigma \le 2^{N-1}\int_b^c\sigma$. Now if $x_k<x<x_{k-1}$, $1<k\le N$, and $y\in I^*$, then

$$\int_x^y \sigma\chi_{(b,c)} = \int_b^y\sigma \le \int_b^c\sigma = 2^{-(k-2)}\int_{x_{k-1}}^{x_{k-2}}\sigma \le 2^{-(k-2)}\int_x^y\sigma.$$

Multiplying both sides of this inequality by $(y-x)^{\alpha-1}$ and taking the sup, we get that for any x, $x_k<x<x_{k-1}$, one has

$$M^+(\sigma\chi_{(b,c)})(x) \le 2^{-(k-2)}M^+(\sigma\chi_{(x,c)})(x), \qquad k=2,\dots,N,$$

while for $k=1$ we have the trivial estimate

$$M^+(\sigma\chi_{(b,c)})(x) \le M^+(\sigma\chi_{(x,c)})(x), \qquad x_1<x<b.$$

Therefore

$$\begin{aligned}\int_a^b (M^+(\sigma\chi_{(b,c)}))^q u &= \sum_1^N \int_{x_k}^{x_{k-1}} (M^+(\sigma\chi_{(b,c)}))^q u\\ &\le \sum_2^N 2^{-(k-2)q}\int_{x_k}^{x_{k-1}} (M^+(\sigma\chi_{(x_k,c)}))^q u + \int_{x_1}^b (M^+(\sigma\chi_{(x_1,c)}))^q u\\ &\le \sum_2^N 2^{-(k-2)q}\int_{x_k}^{c} (M^+(\sigma\chi_{(x_k,c)}))^q u + \int_{x_1}^c (M^+(\sigma\chi_{(x_1,c)}))^q u\\ &\le \sum_2^N 2^{-(k-2)q}\left(\int_{x_k}^c\sigma\right)^{q/p} + \left(\int_{x_1}^c\sigma\right)^{q/p}\\ &\le \left(\sum_2^N 2^{-(k-2)q}2^{kq/p} + 2^{q/p}\right)\left(\int_b^c\sigma\right)^{q/p} \le C\left(\int_b^c\sigma\right)^{q/p}.\end{aligned}$$

3. The case of equal weights

In [1] it is proved that if $1<p\le q$, $q^{-1}=p^{-1}-\alpha$, a necessary and sufficient condition for M_α^+ to be a bounded operator from $L_p(u^p\,dx)$ to $L_q(u^q\,dx)$ is

$(A^+_{p,q})$ There exists C such that for any a and any positive h

$$\left(h^{-1}\int_{(a-h,a)} u^q\right)^{1/q}\left(h^{-1}\int_{(a,a+h)} u^{-p'}\right)^{1/p'} \leq C.$$

We will give a direct proof of the equivalence of this condition with condition $(S^+_{p,q,\alpha,D})$, thus obtaining a geometric proof of this result. Observe that (u^q, u^p) satisfies $(S^+_{p,q,\alpha,D})$ if

$$\left(\int_{(a-h,a+h)}(M^+_\alpha u^{-p'}\chi_{(a,a+h)})^q u^q\right)^{1/q} \leq C\left(\int_{(a,a+h)} u^{-p'}\right)^{1/p},$$

which of course implies

$$\left(\int_{(a-h,a)}(M^+_\alpha u^{-p'}\chi_{(a,a+h)})^q u^q\right)^{1/q} \leq C\left(\int_{(a,a+h)} u^{-p'}\right)^{1/p},$$

but for any x in $(a-h, a)$, $h^{\alpha-1}\int_{(a,a+h)} u^{-p'} \leq M^+_\alpha u^{-p'}\chi_{(a,a+h)}(x)$ and $(A^+_{p,q})$ follows.

To prove the converse we will use the equivalence between $(S^+_{p,q,\alpha,D})$ and $(S^+_{p,q})$ and prove that $(A^+_{p,q})$ implies $(S^+_{p,q,\alpha})$ using the method of [2]. Let I be fixed. A similar argument to the one used in Proposition (2.4) proves that for every x there exists an h (that depends on x) and a constant C_α (that depends only on α) such that

$$M^+_\alpha(\chi_I u^{-p'})(x) \leq C_\alpha h^{\alpha-1}\int_{(x+h/2,x+h)} u^{-p'}.$$

Using $(A^+_{p,q})$ and the relationship between p, q, and α, we obtain

$$\begin{aligned}(M^+_\alpha(\chi_I u^{-p'})(x))^q &\leq C_\alpha h^{\alpha q+p'}\left(\int_{(x,x+h/2)} u^q\right)^{-p'}\\ &= C_\alpha\left(h\left(\int_{(x,x+h/2)} u^q\right)^{-p'/(\alpha q+p')}\right)^{\alpha q+p'}.\end{aligned}$$

Let us now define $s = \alpha q + p'$ and $\beta = 1 - p'/(\alpha q + p')$, and let us consider the operator $M_{u^q,\beta}f(x) = \sup_{x\in I} u^q(I)^{\beta-1}\int_I |f|u^q$. Our last inequality can now be written as

$$(3.1)\qquad (M^+_\alpha(\chi_I u^{-p'})(x))^q \leq C_\alpha(M_{u^q,\beta}(u^{-q}\chi_I))^2.$$

But it is well known [3] that $M_{u^q,\beta}$ maps $L_t(u^q)$ into $L_s(u^q)$, provided $s^{-1} = t^{-1} - \beta$; i.e., $t = s/(1+s\beta)$. Integrating both sides of (3.1) over I and using this result, one gets

$$\begin{aligned}\left(\int_I(M^+_\alpha(\chi_I u^{-p'})(x))^q u^q\right)^{1/q} &\leq C_\alpha\left(\int_I(M_{u^q,\beta}(u^{-q}\chi_I))^s u^q\right)^{1/q}\\ &\leq C_\alpha\left(\int_I u^{-qt}u^q\right)^{s/qt}.\end{aligned}$$

But is easy to check that $t = ps/q$ and $q - qt = -p'$, and therefore, we have proved

$$\left(\int_I (M_\alpha^+(\chi_I u^{-p'})(x))^q u^q\right)^{1/q} \le C_\alpha \left(\int_I u^{-p'}\right)^{1/p},$$

which is $(S_{p,q,\alpha}^+)$.

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