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ON PURELY INFINITE C^*-ALGEBRAS

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ABSTRACT. We find conditions under which a quotient C^*-algebra A/I of a purely infinite C^*-algebra A becomes purely infinite. S. Zhang proved that if A is a σ-unital, simple, purely infinite C^*-algebra, then a hereditary C^*-subalgebra A_x is a stable or unital for each x of A_+. We prove the converse for a completely σ-unital infinite simple C^*-algebra.

INTRODUCTION

A C^*-algebra A is said to be *infinite* if A has an infinite projection and *purely infinite* if the hereditary C^*-subalgebra $(xAx)^-$ contains an infinite projection for each nonzero positive element x of A. Clearly, if A is purely infinite, then A is infinite and every hereditary C^*-subalgebra of A is purely infinite.

In [9], Zhang proved that a purely infinite simple C^*-algebra A has the property **FS**: the set of selfadjoint elements with finite spectra is norm dense in A_{sa}. It is well known [3] that A has **FS** if and only if its real rank $RR(A)$ is zero, that is, the set of invertible selfadjoint elements is norm dense in A_{sa}. Roughly speaking, $RR(A) = 0$ means that A abounds in its projections, so that A satisfies **LP**: A is the closed linear span of its projections, but the converse is not true in general.

There is a purely infinite nonsimple C^*-algebra A with $RR(A) \neq 0$, for example, the multiplier algebra $M(\mathscr{C}\otimes\mathscr{K})$ of $\mathscr{C}\otimes\mathscr{K}$ [11], where $\mathscr{C}$ is the Calkin algebra and $\mathscr{K}$ is the compact operators on the separable infinite-dimensional Hilbert space. It was shown [3, 11] that if A is a C^*-algebra, and if I is a closed two-sided ideal of A, then $RR(A) = 0$ if and only if $RR(I) = 0$, $RR(A/I) = 0$, and every projection in A/I lifts to a projection in A. It is more or less known to experts that a similar result holds for the question of extensions of purely infinite C^*-algebras. In §2, we obtain partial results for the quotients of purely infinite C^*-algebras.

It is not known whether an infinite simple C^*-algebra is purely infinite or not. In §3, we will find a condition for an infinite simple C^*-algebra to be purely infinite.

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Throughout this note, A_x denotes the hereditary C^*-subalgebra $(xAx)^-$ of A generated by a positive element $x \in A$, and $\pi: A \to A/I$ denotes the canonical $*$-homomorphism.

2. Quotients and extensions of purely infinite C^*-algebras

Let I be a closed two-sided ideal in a C^*-algebra A and $\mathrm{ann}(I) = \{a \in A | aI = Ia = 0\}$ the set of all annihilators of I in A. Then $\mathrm{ann}(I)$ becomes a closed two-sided ideal in A. If $\mathrm{ann}(I) = 0$, then A can be regarded as a C^*-subalgebra of the multiplier algebra $M(I)$ of I. If I is a σ-unital simple C^*-algebra with **FS**, then $M(I)/I$ is purely infinite [6, Theorem 1.3], hence A/I also becomes purely infinite whenever A/I sits in $M(I)/I$ as a hereditary C^*-subalgebra. In particular, this happens when A is a hereditary C^*-subalgebra of $M(I)$. In this section, we are interested in the case $\mathrm{ann}(I) \neq 0$.

Theorem 1. *Let A be a purely infinite C^*-algebra and I a closed two-sided ideal in A with $\mathrm{ann}(\mathrm{ann}(I)) = I$. Then the quotient C^*-algebra A/I is purely infinite.*

Proof. For any nonzero positive element x of A, the hereditary C^*-subalgebra $(A/I)_{\pi(x)}$ of A/I generated by $\pi(x)$ is just $\pi(A_x)$ and so it suffices to show that $\pi(A_x)$ has an infinite projection whenever $x \notin I$. First we show that $A_x \cap \mathrm{ann}(I) \neq 0$. Assuming not, for each $b \in (A_x)_+$ and $z \in \mathrm{ann}(I)$, we have $b(z^*z)b \in A_x \cap \mathrm{ann}(I) = 0$, i.e., $zb = 0$. Hence $(A_x)_+ \subseteq \mathrm{ann}(\mathrm{ann}(I)) = I$, which contradicts $\pi(A_x) \neq 0$. Therefore, as a nonzero hereditary C^*-subalgebra of A, $A_x \cap \mathrm{ann}(I)$ has an infinite projection. On the other hand, it is obvious that $\pi(A_x \cap \mathrm{ann}(I)) \cong A_x \cap \mathrm{ann}(I)$. Hence, $\pi(A_x)$ has an infinite projection because $\pi(A_x \cap \mathrm{ann}(I)) \subseteq \pi(A_x)$.

Recall that a C^*-algebra A is said to be *primitive* if A does not contain two orthogonal nonzero ideals.

Corollary 2. *Let A be a purely infinite C^*-algebra and I a closed two-sided ideal in A with $\mathrm{ann}(I) \neq 0$. If A/I is primitive, then A/I is also purely infinite.*

Proof. Assuming $\mathrm{ann}(\mathrm{ann}(I))$ contains I strictly, we have that $\pi(\mathrm{ann}(I))$ and $\pi(\mathrm{ann}(\mathrm{ann}(I)))$ are two orthogonal nonzero ideals in A/I, which contradicts our assumption that A/I is primitive. Hence $\mathrm{ann}(\mathrm{ann}(I)) = I$ and we complete our proof by Theorem 1.

Corollary 3. *Let A be a purely infinite C^*-algebra. If I is a prime closed two-sided ideal in A with $\mathrm{ann}(I) \neq 0$, then A/I is purely infinite.*

Proof. It is easy to show that A/I is primitive if I is prime. Therefore, A/I is purely infinite by Corollary 2.

In [6, Theorem 1.1], it was proved that if A is a σ-unital C^*-algebra with **FS**, then every hereditary C^*-subalgebra of $M(A)$ satisfies **LP**. As mentioned in the introduction, the purely infinite C^*-algebra $M(\mathscr{C} \otimes \mathscr{K})$ does not have **FS**, while every hereditary C^*-subalgebra of $M(\mathscr{C} \otimes \mathscr{K})$ satisfies **LP** since $\mathscr{C} \otimes \mathscr{K}$ is a σ-unital C^*-algebra with **FS**.

Corollary 4. *Let A be a purely infinite C^*-algebra such that every hereditary C^*-subalgebra of A satisfies* **LP**. *If I is a unital closed two-sided ideal in A, then A/I is purely infinite.*

Proof. Let q be the unit of I. To prove our assertion, it suffices to show that $\operatorname{ann}(\operatorname{ann}(I)) = I$ by Theorem 1. Assuming $\operatorname{ann}(\operatorname{ann}(I)) \neq I$, we can take a nonzero projection p in $\operatorname{ann}(\operatorname{ann}(I))$ with $p \notin I$ because I and $\operatorname{ann}(\operatorname{ann}(I))$ satisfy **LP**. Then it is easy to show that $p - pqp$ is a nonzero element of $\operatorname{ann}(I) \cap \operatorname{ann}(\operatorname{ann}(I))$, which is a contradiction.

Remark 5. For the question when purely infiniteness is preserved under extensions, the following partial answer would be well known to the specialists, as was pointed out by the referee: Let I be a closed two-sided ideal in a C^*-algebra A with **FS**. If I and A/I are purely infinite and every projection in A/I lifts to a projection in A, then A is also purely infinite.

3. Stability of purely infinite simple C^*-algebras

Let us recall some terminology: An element x of a unital C^*-algebra A is *well-supported* if there is a projection $p \in A$ with $x = xp$ and x^*x invertible in pAp, and a C^*-algebra A is *completely σ-unital* [7] if every hereditary C^*-subalgebra of A is σ-unital.

Theorem 6. *Let A be a completely σ-unital simple C^*-algebra. Then the following are equivalent*:

(a) *A is purely infinite.*

(b) *A is infinite and A_x is either stable or unital for each nonzero positive element x of A.*

Proof. (a) $\Rightarrow$ (b) This was proved by Zhang [11, Theorem 1.2] for σ-unital purely infinite simple C^*-algebras.

(b) $\Rightarrow$ (a) Let $x \in A_+$. If A_x is stable, then $A_x \cong A_x \otimes \mathscr{K} \cong A \otimes \mathscr{K}$ by [2, Theorem 1.2], and so A_x is infinite because $A \otimes \mathscr{K}$ is infinite by the assumption.

Now, it remains to show that each unital hereditary C^*-subalgebra A_x of A has an infinite projection. If A_x has a positive element y such that A_y is nonunital, then A_y is stable by our assumption, and so it has an infinite projection as above. Therefore, A_x has an infinite projection since $A_y \subseteq A_x$. In order to complete our proof, it suffices to show that every unital hereditary C^*-subalgebra B of A has a positive element y such that A_y, which is a C^*-subalgebra of B, is nonunital. Since $(y^*Ay)^-$ is unital if and only if y is well-supported [1, 6.1], it is equivalent to show that every unital hereditary C^*-subalgebra B of A has an element which is not well-supported.

Assume that every element in a unital hereditary C^*-subalgebra B of A is well-supported. By [1, Theorem 4.3.5], B has **FS**. Hence B has no minimal projection [1, 4] since B is an infinite-dimensional C^*-algebra. Therefore, we can take a sequence of mutually orthogonal projections q_n, $n = 1, 2, 3, \ldots$. Then $y = \sum_{n=1}^{\infty}(2^n)^{-1}q_n$ is a positive element of B, and so there exists a projection $q \in B$ such that $yq = y$ and y^*y is invertible in qBq. Since

$$\frac{1}{2^n}q_n = q_n y = q_n(yq) = \frac{1}{2^n}q_n q,$$

we have $q_n \leq q$ for each $n = 1, 2, 3, \ldots$.

Now, we consider $z_m = y^*y - (2^{2m})^{-1}q$ for $m = 1, 2, 3, \dots$. Then

$$z_m = \sum_{n=1}^{\infty} \frac{1}{2^{2n}} q_n - \frac{1}{2^{2m}}(q_m + (q - q_m))$$
$$= \sum_{n \neq m}^{\infty} \frac{1}{2^{2n}} q_n - \frac{1}{2^{2m}}(q - q_m),$$

and so $q_m z_m = z_m q_m = 0$, $m = 1, 2, 3, \dots$. Hence, z_m is not invertible in qBq, and the spectrum of y^*y in qBq contains the sequence $\{(2^{2m})^{-1} : m = 1, 2, 3, \dots\}$. Therefore the spectrum contains 0, which contradicts the fact that y is well-supported by q.

Remark 7. It is well known [1, Theorem 4.3.5] that a C^*-algebra A has **FS** if and only if the well-supported selfadjoint elements of A are dense in A_{sa}. In spite of this fact, we have shown in the above proof that there is no infinite-dimensional simple unital C^*-algebra with **FS** in which every element is well-supported.

References

1. B. Blackadar, *Comparison theory for simple C^*-algebras*, Operator Algebras and Applications, Vol. 1, London Math. Soc. Lecture Note Ser., vol. 135, Cambridge Univ. Press, London and New York, 1988, pp. 21–54.
2. L. G. Brown, P. Green, and M. A. Rieffel, *Strong Morita equivalence of C^*-algebras*, Pacific J. Math. **71** (1977), 349–363.
3. L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), 131–149.
4. M. Rørdam, *On the structure of simple C^*-algebras tensored with a UHF-algebra*, J. Funct. Anal. **100** (1991), 1–17.
5. J. Tomiyama, *Invitation to C^*-algebras and topological dynamics*, World Scientific, Singapore, 1987.
6. S. Zhang, *On the structure of projections and ideals of corona algebras*, Canad. J. Math. **41** (1989), 721–742.
7. ——, *Stable isomorphism of hereditary C^*-subalgebras and stable equivalence of open projections*, Proc. Amer. Math. Soc. **105** (1989), 677–682.
8. ——, *C^*-algebras with real rank zero and the internal structure of their corona and multiplier algebras, Part* III, Canad. J. Math. **42** (1990), 159–190.
9. ——, *A property of purely infinite simple C^*-algebras* Proc. Amer. Math. Soc. **109** (1990), 717–720.
10. ——, *K_1-groups quasidiagonality and interpolation by multiplier projections*, Trans. Amer. Math. Soc. **325** (1991), 793–818.
11. ——, *Certain C^*-algebras with real rank zero and their corona and multiplier algebras, Part* I, preprint.

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