PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 117, Number 3, March 1993

A SHARP ESTIMATE FOR A^p_α FUNCTIONS IN $\mathbf{C}^n$

DRAGAN VUKOTIĆ

(Communicated by Theodore W. Gamelin)

ABSTRACT. We observe that involutive automorphisms φ_a of the unit ball in $\mathbf{C}^n$ induce surjective involutive isometries of the weighted Bergman space A^p_α $(0 < p < \infty, \alpha > -1)$. By means of these isometries we solve an extremal problem for the point-evaluation functional, thus obtaining a sharp estimate for $|f(z)|$ in terms of $\|f\|_{p,\alpha}$ and $|z|$.

Let $\mathbf{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$ denote the unit ball in $\mathbf{C}^n$, and let ν_n be the normalized Lebesgue measure in $\mathbf{B}^n$. For $0 < p < \infty$, $\alpha > -1$, the *weighted Bergman space* A^p_α is the set of functions f holomorphic in $\mathbf{B}^n$ such that

$$\|f\|_{p,\alpha} = \left\{ \binom{n+\alpha}{n} \int_{\mathbf{B}^n} |f(z)|^p (1-|z|^2)^\alpha \, d\nu_n(z) \right\}^{1/p} < \infty .$$

The presence of the factor $\binom{n+\alpha}{n}$ yields the normalizing condition: $\|\mathbf{1}\|_{p,\alpha} = 1$. This can be verified by an elementary calculation. Namely, let $S_n = \partial \mathbf{B}^n$, and let σ_n be the normalized rotation-invariant positive Borel measure on S_n. Then the following formula for integration in polar coordinates holds (cf. [3, p. 13]):

$$\text{(1)} \qquad \int_{\mathbf{B}^n} u \, d\nu_n = 2n \int_0^1 r^{2n-1} \, dr \int_{S_n} u(r\zeta) \, d\sigma_n(\zeta) .$$

Using this formula and some well-known properties of the beta and gamma functions, we obtain

$$\begin{aligned} \int_{\mathbf{B}^n} (1-|z|^2)^\alpha \, d\nu_n(z) &= 2n \int_0^1 r^{2n-1}(1-r^2)^\alpha \, dr = nB(\alpha+1, n) \\ &= \frac{n\Gamma(\alpha+1)\Gamma(n)}{\Gamma(n+\alpha+1)} = \frac{1}{\binom{n+\alpha}{n}} . \end{aligned}$$

For any $\alpha > -1$, A^p_α is a Banach space for $p \geq 1$ and a metric space for $0 < p < 1$ with the metric given by $d(f, g) = \|f - g\|_p^p$. The subharmonicity

Received by the editors May 13, 1991 and, in revised form, June 26, 1991; presented to the AMS at the Baltimore meeting, January 10, 1992.

1991 *Mathematics Subject Classification*. Primary 30H05; Secondary 46E15.

Key words and phrases. Weighted Bergman space, extremal problem, involutive isometries, point-evaluation functional.

of the function $|f|^p$ for $0 < p < \infty$ (cf. [3, p. 20]), together with a calculation involving formula (1) implies the boundedness of the point-evaluation functional at $a \in \mathbf{B}^n$. Thus it makes sense to consider the extremal problem

$$\sup\{f(a) : f \in A^p_\alpha, \ \|f\|_{p,\alpha} = 1, \ f(a) > 0\}. \tag{2}$$

In this note we prove that there exists a unique extremal function for (2) and we determine it explicitly. As a corollary, we obtain the sharp estimate

$$|f(z)| \leq (1 - |z|^2)^{-(n+\alpha+1)/p} \|f\|_{p,\alpha}, \qquad f \in A^p_\alpha, \ z \in \mathbf{B}^n.$$

Very recently Osipenko and Stessin [2] independently obtained this estimate for $\alpha = 0$, $p \geq 1$ and simply connected domains by making use of reproducing kernels.

We begin by recalling some standard facts (see [3, §2.2]). It is known that the biholomorphic mappings of $\mathbf{B}^n$ onto itself have the form

$$\varphi_a(z) = \frac{a - \frac{\langle z, a\rangle}{|a|^2} a - (1 - |a|^2)^{1/2} \left(z - \frac{\langle z, a\rangle}{|a|^2} a\right)}{1 - \langle z, a\rangle}, \qquad a \in \mathbf{B}^n, \tag{3}$$

up to unitary transformations (for $n = 1$ this is simply the equality $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$).

Observe that $\varphi_a(0) = a$ and φ_a is involutive; i.e.,

$$\varphi_a(\varphi_a(z)) \equiv z, \quad \text{for all } z \in \mathbb{B}^n.$$

Viewing $\mathbf{C}^n$ as $\mathbf{R}^{2n}$, the real Jacobian of φ_a is given by

$$(J_{\mathbf{R}}\varphi_a)(z) = \left(\frac{1 - |a|^2}{|1 - \langle z, a\rangle|^2}\right)^{n+1}, \qquad z \in \mathbf{B}^n. \tag{4}$$

The identities

$$(1 - \langle z, a\rangle)(1 - \langle \varphi_a(z), a\rangle) = 1 - |a|^2 \tag{5}$$

and

$$1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a\rangle|^2} \tag{6}$$

will also be useful.

The following lemma will be the key to our main result.

Lemma. *For $f \in A^p_\alpha$ $(0 < p < \infty, \ \alpha > -1)$ and $a \in \mathbf{B}^n$, the automorphisms φ_a of $\mathbf{B}^n$ given by* (3) *induce surjective isometries*

$$(T_a f)(z) = \left(\frac{1 - |a|^2}{(1 - \langle z, a\rangle)^2}\right)^{(n+\alpha+1)/p} f(\varphi_a(z)), \qquad z \in \mathbf{B}^n, \tag{7}$$

of A^p_α, which are also involutive: $T_a^2 f = f$.

Proof. First we show that T_a is an isometry. The definition (7), the Jacobian formula (4), formula (6), and the change of variable $w = \varphi_a(z)$ yield

$$\begin{aligned} \|T_a f\|^p_{p,\alpha} &= \binom{n+\alpha}{n} \int_{\mathbf{B}^n} |f(\varphi_a(z))|^p \left(\frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a\rangle|^2}\right)^\alpha |J_{\mathbf{R}}\varphi_a(z)| \, d\nu_n(z) \\ &= \binom{n+\alpha}{n} \int_{\mathbf{B}^n} |f(w)|^p (1 - |w|^2)^\alpha \, d\nu_n(w) = \|f\|^p_{p,\alpha}. \end{aligned}$$

Next we show that T_a is surjective. Given $g \in A^p_\alpha$ and $a \in \mathbf{B}^n$, define

$$f(w) = T_a g(w), \tag{8}$$

for $w \in \mathbf{B}^n$. By the above calculation, $f \in A^p_\alpha$. Now recall that φ_a is involutive: $z = \varphi_a(w)$ implies $w = \varphi_a(z)$. Apply the identity (5) to get $g(z) = (T_a f)(z)$. Note that (8) actually shows that T_a is involutive.

Theorem. *An extremal function for* (2) *exists and is unique for* $0 < p < \infty$, $\alpha > -1$. *Furthermore, it is given by*

$$F(z) = (T_a 1)(z) = \left(\frac{1-|a|^2}{(1-\langle z, a\rangle)^2}\right)^{(n+\alpha+1)/p}. \tag{9}$$

Proof. We first consider the case $a = 0$. The subharmonicity of $\log|f|$ implies

$$\log|f(0)| \le \int_{S_n} \log|f(z)|\, d\sigma_n(z)$$

(see [3, p. 20]. Using formula (1) and earlier calculation, we obtain

$$|f(0)| \le \exp\left(\int_{\mathbf{B}^n} \log|f(z)|\, d\mu_n(z)\right),$$

where

$$d\mu_n(z) = \binom{n+\alpha}{n}(1-|z|^2)^\alpha\, d\nu_n(z)$$

is a unit measure on $\mathbf{B}^n$. On the other hand, the geometric-arithmetic mean inequality applied to $|f|^p$, $0 < p < \infty$ (cf. [1, §6.7]), yields

$$\exp\left(\int_{\mathbf{B}^n} \log|f(z)|^p\, d\mu_n(z)\right) \le \int_{\mathbf{B}^n} |f(z)|^p\, d\mu_n(z),$$

hence

$$\exp\left(\int_{\mathbf{B}^n} \log|f(z)|\, d\mu_n(z)\right) \le \|f\|_{p,\alpha}.$$

Thus, for our problem (2), $f(0) \le \|f\|_{p,\alpha}$ and equality holds if and only if f is constant and positive, hence $f \equiv 1$ is the unique extremal function.

For arbitrary $a \in \mathbf{B}^n$, the lemma implies

$$\{f \in A^p_\alpha : \|f\|_{p,\alpha} = 1,\ f(a) > 0\} = \{f \in A^p_\alpha : \|T_a f\|_{p,\alpha} = 1,\ (T_a f)(0) > 0\}.$$

Thus a function f is extremal for point-evaluation at a if and only if $T_a f$ is extremal for point-evaluation at 0. From the case $a = 0$ we infer that there is a unique extremal function F that satisfies $(T_a F)(z) \equiv 1$. Since T_a is involutive, (9) follows.

Corollary. *For any* $f \in A^p_\alpha$ *and for every* $z \in \mathbf{B}^n$,

$$|f(z)| \le (1-|z|^2)^{-(n+\alpha+1)/p}\|f\|_{p,\alpha}. \tag{10}$$

For each point $a \in \mathbf{B}^n$, *the equality in* (10) *is attained at that point only for functions of the form* $f(z) = \lambda(T_a 1)(z)$, *with* $\lambda \in \mathbf{C}$ *arbitrary.*

Acknowledgments

The results in this note constitute part of the author's Ph.D. thesis at the University of Michigan under the supervision of Professor Peter L. Duren. The author thanks Sheldon Axler for a useful discussion on this problem and the referee for valuable suggestions.

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MATEMATIČKI FAKULTET, STUDENTSKI TRG 16/IV, 11000 BEOGRAD, YUGOSLAVIA

Current address: Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

E-mail address: dragan@math.lsa.umich.edu