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A NOTE ON NONLINEAR VOLTERRA OPERATORS

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ABSTRACT. In this paper we discuss a class of nonlinear Volterra operators and show that they are globally invertible.

1. INTRODUCTION

In this paper we study nonlinear Volterra operators of the form

$$A: u \mapsto u - \int_0^x k(x, y) f(u(y))\, dy .$$

These operators are regarded as mappings between function spaces and, hence, the abstract results of Functional Analysis may be brought to bear on these mappings with good results. Since linear Volterra operators are invertible, the nonlinear Volterra operators we consider are locally invertible. Global hypotheses suffice to ensure that these operators are globally invertible. More specifically, the operator A is a Fredholm operator of index zero. Suitable hypotheses on $f: R \to R$ ensure that A is a proper mapping. A well-known result regarding nonlinear Fredholm operators (see below) enables us to show that A is globally invertible.

We do the simple case here; namely, the operator A is defined on the space of continuous functions over the interval $[0, 1]$. There is no loss in generality since the results are easily carried over, with suitable modifications, to operators defined on the L_p spaces.[1]

2. RESULTS

We begin by recalling some definitions and results that will be used in the sequel.

Definition 1. Let $A: E \to E$ be a continuous mapping on a Banach space E. We say A is a proper mapping if the inverse image $A^{-1}(S)$ of every compact set S in E is compact.

We now state a theorem regarding Fredholm operators. The proof of this theorem may be found in Berger [3]. The statement of the theorem requires

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[1]See Hochstadt or Zaanen for the theory of Volterra operators.

some terminology. Let $A: E \to E$ be a C^1 mapping on a Banach space E. A point u is a singular point of A if $A'(u)$, the derivative of A at u, fails to be invertible. A singular value is the image of a singular point. Let Σ_p denote the singular points of A, and let $\Sigma_v = A(\Sigma_p)$ denote the singular values of A.

Theorem 1. *Suppose $A: E \to E$ is a C^1 proper Fredholm operator of index zero. Then the cardinality of the set $A^{-1}(w)$ is finite and constant on connected components of $E - \Sigma_v$. In particular, if Σ_p is empty then A is a C^1 global diffeomorphism.*

We now consider a class of nonlinear Volterra operators and show that they satisfy the conditions of the above theorem.

Let E denote the space of continuous functions on the unit interval $[0, 1]$, equipped with the sup-norm $\|\cdot\|$. Let $k(x, y)$ denote a continuous kernel on the unit square and let $f: R \to R$ be a C^1 function. Define $A: E \to E$ by $A(u) = u - Kf(u)$, where $K: h \mapsto \int_0^x k(x, y)h(y)\,dy$. Since f is C^1, the Omega Lemma[2] implies that the operator $N: u \mapsto f(u)$ is continuously differentiable. Since K is linear and bounded, $A = I - KN$ is C^1.

The derivative of A at a point u of E is given by the linear operator

$$A'(u): h \mapsto h - K(f'(u)h).$$

So $A'(u)$ is of the form identity + compact. Hence, A is a Fredholm operator of index 0. Since $A'(u)$ is a linear Volterra operator, its null space is always trivial. Consequently, by the inverse function theorem, A is a C^1 local diffeomorphism.

Next, suppose $f(x)/x \to M_2$ as $x \to +\infty$ and $f(x)/x \to M_1$ as $x \to -\infty$. Assume $f(0) = 0$. With these hypotheses on f, we show that A is proper.

Theorem 2. *$A: E \to E$ is proper and, hence, a C^1 global diffeomorphism.*

Proof. We adapt the proof in [2] to show that A is proper. The idea behind the proof is to obtain a linear homogenous equation at infinity and then obtain a contradiction.

Let S be a compact set in E and let $\{u_n\}$ be a sequence in $A^{-1}(S)$. Let $v_n = A(u_n) = u_n - K(f(u_n))$. Since S is compact, a subsequence of $\{v_n\}$ converges in S. If $\{u_n\}$ is bounded then the sequence $\{f(u_n)\}$ is bounded, and since K is compact, a subsequence of $\{K(f(u_n))\}$ converges. Hence, a subsequence of $\{u_n\}$ converges. Since $A^{-1}(S)$ is closed the limit of this subsequence must lie in $A^{-1}(S)$. Thus to show that A is proper it is sufficient to show that the sequence $\{u_n\}$ is bounded.

We argue by contradiction. Suppose $\{u_n\}$ is unbounded. Then passing to a subsequence if necessary we have $\|u_n\| \to \infty$. Let $z_n = u_n/\|u_n\|$, and let

$$h(y) = \begin{cases} f(y)/y & \text{if } y \neq 0, \\ f'(0) & \text{otherwise}. \end{cases}$$

Then $v_n/\|u_n\| = z_n - K(h(u_n)z_n)$. Since a subsequence of $\{v_n\}$ converges and z_n and $h(u_n)$ are bounded, the compactness of K implies that a subsequence of $\{z_n\}$ converges, say to z, in E. For notational convenience, we rename this subsequence as $\{z_n\}$. Thus, $z_n \to z$.

[2]See Abraham, Marsden, and Ratiu.

Next, let

$$w(x) = \begin{cases} M_2 & \text{if } z(x) > 0, \\ M_1 & \text{if } z(x) > 0, \\ f'(0) & \text{if } z(x) = 0. \end{cases}$$

Now $z(x) > 0$ implies $\lim_{n\to+\infty} u_n(x) = +\infty$ and $z(x) < 0$ implies

$$\lim_{n\to+\infty} u_n(x) = -\infty .$$

Thus, $\lim_{n\to+\infty} h(u_n(x)) = M_2$ when $z(x) > 0$ and $\lim_{n\to+\infty} h(u_n(x)) = M_1$ when $z(x) < 0$. Hence,

$$\lim_{n\to+\infty} k(x, y)h(u_n(y))z_n(y) = k(x, y)w(y)z(y).$$

Furthermore, for fixed x and each n, $|k(x, y)h(u_n(y))z_n(y)|$ is dominated by an absolutely integrable function. Hence, by Lebesgue's theorem, we have

$$0 = \lim_{n\to+\infty} \frac{v(x)}{\|u_n\|} = \lim_{n\to+\infty} z_n - \lim_{n\to+\infty} K(h(u_n)z_n) = z - K(wz).$$

Since $z_n \to z$ and $\|z_n\| = 1$, we have $\|z\| = 1$. Clearly, $\int_0^1 \|z\|^2 > 0$. But this implies that the linear Volterra operator $g \mapsto g - K(wg)$, defined on $L_2[0, 1]$, has a nontrivial nullspace, which is impossible. Thus, the sequence $\{u_n\}$ must be bounded. Hence the operator A is proper as claimed. In view of Theorem 1, we conclude that A is a global homeomorphism. This completes the proof.

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