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SPACES WHOSE nTH POWER IS WEAKLY INFINITE-DIMENSIONAL BUT WHOSE $(n+1)$TH POWER IS NOT

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ABSTRACT. For every natural number n we construct a metrizable separable space Y such that Y^n is weakly infinite-dimensional (moreover, is a C-space) but Y^{n+1} is strongly infinite-dimensional.

1. INTRODUCTION

All our spaces are metrizable separable. Our terminology follows [3]. The symbol P denotes the space of irrational numbers and N the space of natural numbers. A space X is weakly infinite-dimensional (w.i.d.) if for every sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of X there exist closed sets $L_1, L_2, \ldots$ such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{\infty} L_i = \varnothing$. A space is strongly infinite-dimensional (s.i.d.) if it is not w.i.d.

A space X is called a C-space (or has property C) if for every sequence $\mathscr{G}_1, \mathscr{G}_2, \ldots$ of open covers of X there exists a sequence $\mathscr{U}_1, \mathscr{U}_2, \ldots$ of families of pairwise disjoint open subsets of X, the union of which covers X, such that each member of $\mathscr{U}_i$ is contained in a member of $\mathscr{G}_i$ (see, for example, [3, §8] or [18] for this notion). Every countable-dimensional space has property C, and every space with property C is weakly infinite-dimensional.

In [14] an example was given showing that the square of a w.i.d. space may be s.i.d., and in [12] we have constructed a C-space whose product with a 0-dimensional space is s.i.d. (under the Continuum Hypothesis, the 0-dimensional factor may be the space of irrationals P).

In this note we show that for every n the assumption that the nth power Y^n of Y is a C-space may not prevent Y^{n+1} from being s.i.d. In fact, we show that each complete C-space which behaves regularly with respect to finite products, but is not hereditarily w.i.d., is a source of such examples.

Theorem 1. *Let X be a completely metrizable space such that*
(i) *for each C-space E, the product $X \times E$ is a C-space,*
(ii) *X contains a strongly infinite-dimensional subspace.*

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Then for each $n \in N$ *there exists* $Y \subset P \times X$, P *being the irrationals, such that*

(a) Y^n *is a* C*-space (hence is weakly infinite-dimensional),*
(b) Y^{n+1} *is strongly infinite-dimensional.*

Theorem 2. *Under the assumptions of Theorem* 1 *there exists* $Y \subset P \times X$ *such that* Y^n *is a* C*-space for every* $n \in N$ *but the product of* Y *with a certain subspace of irrationals is strongly infinite-dimensional.*

Note that the weakly infinite-dimensional compactum which is not countable-dimensional, constructed by R. Pol in [13], satisfies conditions (i) and (ii) (see [18, §3, first Corollary] for the proof of property (i)).

Garity, Hattori, Rohm, and Yamada (see [4, 5, 6, 18, 19]) have shown that in the presence of σ-compactness the products of spaces with property C behave in a regular way. Our examples, based on totally imperfect sets, lack compactness-like properties. Let us point out, however, that, as demonstrated in [15], under the Continuum Hypothesis even the Menger Property, a covering property close to σ-compactness, does not exclude the irregularities exhibited in Theorem 1.

The construction given in this paper improves our earlier construction [12, Example 1] by replacing the classical Bernstein's sets with their modifications considered by van Douwen [2] and Przymusiński [16]. We decided to include, for the reader's convenience, a simple construction of such sets in a form suitable for our purposes. As indicated by R. Pol (see [16, Remark 3]), this can be achieved by arguments due to Mycielski [11] and Kuratowski [8], concerning independent perfect sets.

Some basic ideas of the construction we apply go back to Michael [9, 10] (and were subsequently developed by Alster and Zenor [1], van Douwen [2], and Przymusiński [17]).

2. Auxiliary lemmas

The lemma below is a particular case of results of Kuratowski [8] and Mycielski [11] and the proof we give is an adaptation of their arguments to our special situation.

Lemma 1. *Let* $\mathscr{F}$ *be a countable family of continuous maps* $f : S \to T$ *from a completely metrizable space* S *to a Hausdorff space* T. *If* S *contains a subset* A *without isolated points such that each* $f \in \mathscr{F}$ *is injective on* A, *then* S *contains a perfect set with this property.*

Proof. Let $Z = \operatorname{cl} A$ and let $\mathscr{K}(Z)$ be the space of all compact subsets of Z with the Hausdorff metric. Let $\mathscr{G} = \{K \in \mathscr{K}(Z) \colon \text{all } f \in \mathscr{F} \text{ are injective on } K\}$. The set $\mathscr{G}$ is a G_δ-set in $\mathscr{K}(Z)$ and, since the finite subsets of A form a subset of $\mathscr{G}$ dense in $\mathscr{K}(Z)$, $\mathscr{G}$ is residual. Now, since the family of perfect subsets of Z is residual in $\mathscr{K}(Z)$ (cf. [8, Proposition 2]), $\mathscr{G}$ contains a perfect set.

The following result was essentially proved in [2, 16].

Lemma 2. *Let* $\mathscr{F}$ *be a countable family of continuous maps* $f : S \to T$ *from a completely metrizable space* S *to a Hausdorff space* T. *Then there exists a*

countable collection $\mathscr{D}$ of subsets of S such that

(a) *the sets $\bigcup\{f(D) : f \in \mathscr{F}\}$, for $D \in \mathscr{D}$, are pairwise disjoint,*
(b) *if $D \subset U$ with $D \in \mathscr{D}$ and U open in S, then for some countable $E \subset T$ we have $S \backslash U \subset \bigcup\{f^{-1}(E) : f \in \mathscr{F}\}$.*

Proof. We repeat the classical Bernstein's argument. Let $\{F_\alpha : \alpha < 2^\omega\}$ be the collection of all perfect sets in S on which every $f \in \mathscr{F}$ is injective (if this collection is empty, then put $\mathscr{D} = \{\varnothing, \varnothing, \dots\}$). Let us choose inductively (with respect to pairs (i, α)) points $x_i^\alpha \in F_\alpha$ with $f(x_i^\alpha) \neq g(x_j^\beta)$, if $(i, \alpha) \neq (j, \beta)$, where $i, j \in N$, $\alpha, \beta < 2^\omega$, $f, g \in \mathscr{F}$, and let us set $D_i = \{x_i^\alpha : \alpha < 2^\omega\}$, $\mathscr{D} = \{D_i : i = 1, 2, \dots\}$. Suppose that $D_i \subset U$, where U is open in S, and let M be a maximal subset of $S \backslash U$ such that each $f \in \mathscr{F}$ is injective on M. By Lemma 1, the set M is dispersed, hence countable, and (b) follows with $E = \bigcup\{f(M) : f \in \mathscr{F}\}$.

The next lemma is due to Rubin [20]. It is not, however, stated explicitly in [20] and, instead of referring the reader to some details of Rubin's proof, we decided to indicate a standard argument needed to derive the fact we need from a theorem formulated in [20].

Lemma 3. *Every strongly infinite-dimensional space X contains a strongly infinite-dimensional totally disconnected subspace.*

Proof. Indeed, as proved by Rubin [20, Theorem 3.1], every s.i.d. space X contains an s.i.d. subspace Y all of whose subspaces are either 0-dimensional or strongly infinite-dimensional. On the other hand, each space Y of dimension ≥ 2 contains a totally disconnected subspace of positive dimension. This can be justified as follows: let A and B be disjoint closed subsets of Y such that each partition in Y between A and B has positive dimension and let $f : Y \to [0, 1]$ be a continuous mapping with $f^{-1}(0) = A$ and $f^{-1}(1) = B$. By Hilgers' argument (see [7, Chapter II, §27, IX, Theorem]) there exists $M \subset Y$ with $f(M)$ irrationals (in fact, $M \cap f^{-1}(t)$ is a singleton for each irrational t) such that each G_δ-set containing M contains also some $f^{-1}(t)$, where $t \in (0, 1)$. Then M is totally disconnected, but by the enlargement theorem (see [7, Chapter II, §27, IV, Theorem 1]) M is not 0-dimensional as $\dim f^{-1}(t) \geq 1$ for each $t \in (0, 1)$.

3. Proofs

Proof of Theorem 1 (see [3] for facts about weakly infinite-dimensional spaces). Suppose that X is a space satisfying conditions (i) and (ii) of Theorem 1. Set $Z = P \times X$ and let $p : Z \to P$ be the projection. For $j \leq m$ let $p_j : Z^m \to P$ be the composition of the jth projection with p. Put $S = \bigoplus_{m=1}^\infty Z^m$ and let $f_j : S \to P$ be such that $f_j(x)$ is $p_j(x)$ if $x \in Z^m$ with $m \geq j$ or $p_1(x)$ otherwise.

Let $D_1, D_2, \dots$ be the sets satisfying conditions (a) and (b) of Lemma 2 for S, $T = P$, and $\mathscr{F} = \{f_1, f_2, \dots\}$. Then for every $i \in N$ the sets $B_i = \bigcup_{j=1}^\infty f_j(D_i)$ are disjoint totally imperfect subsets of P such that if $A_i =$

$B_i \times X \subset Z$, then

(1) if $m \in N$ and $U \subset Z^m$ is open and contains A_i^m then $Z^m \backslash U$ is contained in countably many fibers $p_j^{-1}(d)$, where $d \in P$ and $j \leq m$.

This follows directly from conditions (a) and (b) of Lemma 2.

Since the space X is not hereditarily w.i.d., by Lemma 3 there exists an s.i.d. subset M of X which is totally disconnected. Let $h : M \to P$ be an injection (see [7, Chapter V, §46, V, Theorem 3]); then the set

$$\widetilde{M} = \{(h(x), x) : x \in M\} \subset Z$$

is homeomorphic to M.

Let us fix $n \in N$. For every $i \leq n+1$ let $C_i = \bigcup\{A_j : j \neq i,\ j \leq n+1\}$ and let $Y_i = C_i \cup \widetilde{M}$ be the subspace of Z.

We will show that the space $Y = \bigoplus_{i=1}^{n+1} Y_i$ satisfies conditions (a) and (b) of Theorem 1.

To prove (a) it suffices to show that for every $k \leq n$ and every $i_1, \dots, i_k \in \{1, \dots, n+1\}$

(2) the Cartesian product $Y_{i_1} \times \cdots \times Y_{i_k}$ has property C.

Suppose that $m = 1$ or (2) is true for $k < m$, where $m \leq n$. Put $K = Y_{i_1} \times \cdots \times Y_{i_m}$. To prove that K is a C-space it suffices to show that

(3) if $\mathscr{G}_1, \mathscr{G}_2, \dots$ is a sequence of open covers of K then there exists a pairwise disjoint open refinement $\mathscr{U}_n$ of $\mathscr{G}_n$ such that $\bigcup\{\mathscr{U}_n : n \in N\}$ covers K.

Since $m \leq n$, there exists $j \in \{1, \dots, n+1\} \backslash \{i_1, \dots, i_m\}$. We have $A_j \subset C_{i_l}$ for $l = 1, \dots, m$, thus $A_j^m \subset K$. Since A_j^m is the Cartesian product of X^m with a subset B_j^m of P^m and since X satisfies (i), the space A_j^m is a C-space. Thus, for every $p = 1, 2, \dots$ there exists a family $\mathscr{W}_{2p-1}$ consisting of disjoint open subsets of Z^m such that $\mathscr{U}_{2p-1} = \{W \cap K : W \in \mathscr{W}_{2p-1}\}$ is a refinement of $\mathscr{G}_{2p-1}$ and the set $U = \bigcup \bigcup_{p=1}^{\infty} \mathscr{W}_{2p-1}$ contains A_j^m. By (1), the set $Z^m \backslash U$ is contained in countably many fibers of the form $H = \{x \in Z^m : p_k(x) = d\}$, where $k \leq m$ and $d \in D \subset P$, with $|D| \leq \aleph_0$. If $m > 1$ then for every such H the set $H \cap K$ is closed in K and is homeomorphic with $\mathbb{P}_{l \neq k} Y_{i_l} \times X$, or with $\mathbb{P}_{l \neq k} Y_{i_l} \times \{(d, h^{-1}(d))\}$, if $d \in h(M) \backslash (\bigcup\{B_j : j \neq i_k,\ j \leq n+1\})$, or it is empty. By the inductive assumption and (i) the set $H \cap K$ is a C-space. If $m = 1$, then $H \cap K$ is homeomorphic to X, one-point, or the empty space, hence it is a C-space. In both cases the set $K \backslash U$ is contained in countably many closed subsets of K of the form $H \cap K$ with property C, hence it has property C. Thus for every $p = 1, 2, \dots$ there exists a pairwise disjoint open refinement $\mathscr{U}_{2p}$ of $\mathscr{G}_{2p}$ such that $\bigcup_{p=1}^{\infty} \mathscr{U}_{2p}$ covers $K \backslash U = K \backslash \bigcup_{p=1}^{\infty} \mathscr{U}_{2p-1}$. The families $\mathscr{U}_1, \mathscr{U}_2, \dots$ satisfy the required conditions, which finishes the proof of (3).

We will show now that Y^{n+1} is strongly infinite-dimensional. It suffices to prove that $Y_1 \times Y_2 \times \cdots Y_{n+1}$ is s.i.d. Let

$$\Delta = \{x = (x_1, \dots, x_{n+1}) \in Y_1 \times \cdots \times Y_{n+1} : x_1 = \cdots = x_{n+1}\}.$$

Let $x = (z, z, \dots, z) \in \Delta$. If $z \notin \widetilde{M}$, then $z \in A_j$ for some $j \leq n+1$ and at the same time $z \in Y_j \backslash \widetilde{M} \subset C_j$, contrary to $C_j \cap A_j = \varnothing$. Thus $z \in \widetilde{M}$, which implies that Δ is homeomorphic with $\widetilde{M}$. Thus $Y_1 \times \cdots \times Y_{n+1}$ is s.i.d., since it contains a closed subspace which is s.i.d.

Proof of Theorem 2. Let $B_1, B_2, \dots$ be the sets constructed in the proof of Theorem 1. Then the sets $B_1' = B_1$ and $B_2' = P \backslash B_1'$ form a decomposition of P into two disjoint Bernstein sets such that condition (1) is satisfied for $A_i = B_i' \times X$, $i = 1, 2$. Let M, $h: M \to P$, and $\widetilde{M}$ be as in the proof of Theorem 1 and let

$$\widetilde{M}_i = \{(h(x), x) : x \in h^{-1}(B_i')\}$$

for $i = 1, 2$. Since $\widetilde{M} = \widetilde{M}_1 \cup \widetilde{M}_2$ is s.i.d., then either $\widetilde{M}_1$ or $\widetilde{M}_2$ is s.i.d. Suppose that $\widetilde{M}_1$ is s.i.d. and let $Y = \widetilde{M}_1 \cup A_2$.

We will show that

(4) the product $Y \times B_1'$ of Y with a subspace B_1' of P is s.i.d.

Similarly as in [12, Example 1], let $f : Y \to P$ be the projection; then $f^{-1}(B_1') = \widetilde{M}_1$ and $\mathrm{Graph}(f|\widetilde{M}_1)$ is homeomorphic to $\widetilde{M}_1$, so it is s.i.d. This implies that $Y \times B_1'$ is s.i.d. (see [12, Lemma 2]).

It remains to check that

(5) Y^n is a C-space for every $n \in N$.

This can be proved inductively, similarly to condition (2), using the facts that $Y^n \supset A_2^n$ for every $n \in N$ and A_2 satisfies (1).

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