

ON TWO ABSOLUTE RIESZ SUMMABILITY FACTORS OF INFINITE SERIES

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ABSTRACT. This paper gives a necessary and sufficient condition in order that a series $\sum a_n \varepsilon_n$ should be summable $|R, q_n|$ whenever $\sum a_n$ is summable $|R, p_n|_k$, $k \geq 1$, and so extends the known result of Bosanquet to the case $k > 1$.

1. INTRODUCTION

Let $\sum a_n$ be a given infinite series with s_n its n th partial sum, and let (p_n) be a sequence of positive real numbers such that $P_n = p_0 + p_1 + \cdots + p_n \rightarrow \infty$ as $n \rightarrow \infty$. If

$$\sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^k < \infty,$$

where t_n denotes Riesz mean of $\sum a_n$, i.e.,

$$t_n = P_n^{-1} \sum_{v=0}^n p_v s_v,$$

then the series $\sum a_n$ is said to be summable $|R, p_n|_k$, $k \geq 1$. Also $|R, p_n| \equiv |R, p_n|_1 \equiv |\bar{N}, p_n|$.

2. MAIN RESULT

It is known that the summability $|R, q_n|$ and the summability $|R, p_n|_k$, $k \geq 1$, are in general independent of each other. It is, therefore, natural to find out suitable summability factors (ε_n) so that $\sum a_n \varepsilon_n$ may be summable $|R, p_n|_k$ whenever $\sum a_n$ is summable $|R, q_n|$, and, conversely, if $\sum a_n$ is summable $|R, p_n|_k$ then $\sum a_n \varepsilon_n$ may be summable $|R, q_n|$. In a paper [4] the author has examined the summability factors problem of the first type. We propose to study the converse problem in the present paper. In what follows we shall prove the following

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Theorem. *The necessary and sufficient condition for a series $\sum a_n \varepsilon_n$ to be summable $|R, q_n|$ whenever $\sum a_n$ is summable $|R, p_n|_k, k \geq 1$, are*

$$\left\{ m^{-1/k'} \left(\frac{q_m P_m |\varepsilon_m|}{Q_m p_m} + \left| \frac{P_m \Delta \varepsilon_m}{p_m} + \varepsilon_{m+1} \right| \right) \right\} \in l_{k'},$$

where $1/k + 1/k' = 1$.

It may be remarked that our theorem includes, as a special case $k = 1$ and $\varepsilon_m = 1$, the following theorem of Sunouchi [5] and Bosanquet [2].

Theorem A. *The necessary and sufficient condition for a series $\sum a_n$ to be summable $|\bar{N}, q_n|$ whenever it is summable $|\bar{N}, p_n|$ is*

$$q_n P_n / Q_n p_n = O(1) \text{ as } n \rightarrow \infty.$$

The sufficiency of this theorem was proved by Sunouchi [5].

3. REQUIRED LEMMAS

We require the following lemmas for the proof of the theorem.

Lemma 1. *Let $p \geq 1, k \geq 1$ and suppose that x, y, u , and v are related as*

$$y_n = \sum_{m=0}^{\infty} c_{nm} x_m, \quad n \geq 0,$$

$$v_m = \sum_{n=0}^{\infty} c_{nm} u_n, \quad m \geq 0.$$

The necessary and sufficient condition for $y \in l_p$ whenever $x \in l_k$ is $v \in l_{k'}$ whenever $u \in l_{p'}$, where k' and p' are the conjugate indices of k and p , respectively (see [3, Lemma 14]).

Lemma 2. *If $k > 1$ and $y_n = \sum_{m=0}^n c_{nm} x_m$ and $\sum_{n=0}^{\infty} |y_n| < \infty$ whenever $\sum_{m=0}^{\infty} |x_m|^k < \infty$, then $\sum_{n=0}^{\infty} |c_{nn}|^{k'} < \infty$.*

This lemma is contained in the corollary to Theorem 10 of [1].

4. PROOF OF THE THEOREM

Since the proof is easy for $k = 1$, it is omitted. Suppose that $k > 1$, and let (t_n) and (T_n) be the sequences of Riesz means (R, p_n) and (R, q_n) of the series $\sum a_n$ and $\sum a_n \varepsilon_n$, respectively. Then by the definition, we have

$$t_n = \frac{1}{P_n} \sum_{v=0}^n p_v \sum_{r=0}^v a_r = \frac{1}{P_n} \sum_{v=0}^n (P_n - P_{v-1}) a_v, \quad P_{-1} = 0,$$

$$T_n = \frac{1}{Q_n} \sum_{v=0}^n q_v \sum_{r=0}^v a_r \varepsilon_r = \frac{1}{Q_n} \sum_{v=0}^n (Q_n - Q_{v-1}) a_v \varepsilon_v, \quad Q_{-1} = 0.$$

A few calculations reveal that, for $n \geq 1$,

$$(4.1) \quad x_n = t_n - t_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n P_{v-1} a_v, \quad x_0 = a_0,$$

and

$$(4.2) \quad y_n = T_n - T_{n-1} = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^n Q_{v-1} a_v \varepsilon_v, \quad y_0 = a_0 \varepsilon_0.$$

It follows by making use of (4.1) that

$$\begin{aligned} y_n &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^n Q_{v-1} \varepsilon_v \left(\frac{P_v}{p_v} x_v - \frac{P_{v-2}}{p_{v-1}} x_{v-1} \right) \\ &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} Q_{v-1} \varepsilon_v \frac{P_v}{p_v} x_v + \frac{q_n P_n \varepsilon_n}{Q_n p_n} x_n \\ &\quad - \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^n Q_{v-1} \varepsilon_v \frac{P_{v-2}}{p_{v-1}} x_{v-1} \\ &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \Delta(Q_{v-1} \varepsilon_v) + Q_v \varepsilon_{v+1} \right) x_v + \frac{q_n P_n \varepsilon_n}{Q_n p_n} x_n. \end{aligned}$$

Put $X_n = n^{1-1/k} x_n$ for $n \geq 1$. Then

$$\begin{aligned} y_n &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \Delta(Q_{v-1} \varepsilon_v) + Q_v \varepsilon_{v+1} \right) v^{1/k-1} X_v + \frac{q_n P_n \varepsilon_n}{Q_n p_n} n^{1/k-1} X_n \\ &= \sum_{v=1}^{\infty} c_{nv} X_v, \end{aligned}$$

where

$$c_{nv} = \begin{cases} \frac{q_n}{Q_n Q_{n-1}} \left(\frac{P_v}{p_v} \Delta(Q_{v-1} \varepsilon_v) + Q_v \varepsilon_{v+1} \right) v^{1/k-1}, & 1 \leq v \leq n-1, \\ \frac{q_n P_n \varepsilon_n}{Q_n p_n} n^{1/k-1}, & v = n, \\ 0, & v > n. \end{cases}$$

Now $\sum a_n \varepsilon_n$ is summable $|R, q_n|$ whenever $\sum a_n$ is summable $|R, p_n|_k$ if and only if

$$(4.3) \quad \sum |y_n| < \infty \quad \text{whenever} \quad \sum |X_n|^k < \infty.$$

Using Lemma 1, the necessary and sufficient conditions for the same are

$$\sum_{n=v}^{\infty} c_{nv} u_n \text{ is convergent for every } u_n = O(1), \quad v \geq 1,$$

and

$$\sum_{v=1}^{\infty} \left| \sum_{n=v}^{\infty} c_{nv} u_n \right|^{k'} < \infty \quad \text{whenever } u_n = O(1).$$

Now

$$\begin{aligned} \sum_{n=v}^{\infty} c_{nv} u_n &= \frac{q_v P_v \varepsilon_v}{Q_v p_v} v^{1/k-1} u_v + \left(\frac{P_v}{p_v} \Delta(Q_{v-1} \varepsilon_v) + Q_v \varepsilon_{v+1} \right) v^{1/k-1} \sum_{n=v+1}^{\infty} \frac{q_n u_n}{Q_n Q_{n-1}} \\ &= \frac{q_v P_v \varepsilon_v}{Q_v p_v} v^{1/k-1} u_v + \left(\frac{P_v}{p_v} \Delta(Q_{v-1} \varepsilon_v) + Q_v \varepsilon_{v+1} \right) v^{1/k-1} \delta_v, \end{aligned}$$

where $\delta_v = \sum_{n=v+1}^{\infty} (q_n/Q_n Q_{n-1}) u_n$, which is convergent, since $u_n = O(1)$. Thus, a necessary and sufficient condition for (4.3) is

$$(4.4) \quad \sum_{v=1}^{\infty} \left| \frac{q_v P_v \varepsilon_v}{Q_v p_v} v^{1/k-1} u_v + \left(\frac{P_v}{p_v} \Delta(Q_{v-1} \varepsilon_v) + Q_v \varepsilon_{v+1} \right) v^{1/k-1} \delta_v \right|^{k'} < \infty,$$

whenever $u_n = O(1)$.

Necessity. Let $u_v = 1$ for all $v \geq 1$. Then $\delta_v = 1/Q_v$. Hence, it follows from (4.4) that

$$(4.5) \quad \sum_{v=1}^{\infty} \left\{ v^{-1/k'} \left| \frac{P_v}{p_v} \Delta \varepsilon_v + \varepsilon_{v+1} \right| \right\}^{k'} < \infty.$$

In addition, we have, by Lemma 2,

$$(4.6) \quad \sum_{n=1}^{\infty} \left\{ n^{-1/k'} \left| \frac{q_n P_n \varepsilon_n}{Q_n p_n} \right| \right\}^{k'} < \infty.$$

Hence, combining (4.5) and (4.6), we get from the inequality

$$a^{k'} + b^{k'} \leq (a + b)^{k'} \leq 2^{k'} (a^{k'} + b^{k'}), \quad a, b \geq 0,$$

that the hypothesis of the theorem is necessary.

Sufficiency. This part of the proof follows immediately from (4.5), (4.6), and the above inequality, since (4.4) holds whenever $u_n = O(1)$. Therefore the proof of the theorem is completed.

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