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EULERIAN OPERATORS AND THE JACOBIAN CONJECTURE

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ABSTRACT. In this paper we introduce a new class of polynomial maps, the so-called nice polynomial maps. Using Eulerian operators we show how for these polynomial maps the main results obtained by Bass (*Differential structure of étale extensions of polynomial algebras*, Proc. Workshop on Commutative Algebra, MSRI, 1987) can be proved in a very simple and elementary way. Furthermore we show that every polynomial map F satisfying the Jacobian condition, $\det JF \in k^*$, is equivalent to a nice polynomial map; more precisely the polynomial map $F_{(\lambda)}(X) = F(X+\lambda) - F(\lambda)$ is nice for almost all $\lambda \in k^n$.

INTRODUCTION

Let $F = (F_1, \dots, F_n)\colon \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map, i.e., each coordinate function F_i belongs to the polynomial ring $\mathbb{C}[X] := \mathbb{C}[X_1, \dots, X_n]$. The Jacobian Conjecture asserts that $\det JF \in \mathbb{C}^*$ implies that F is invertible, i.e., $\mathbb{C}[X] = \mathbb{C}[F] := \mathbb{C}[F_1, \dots, F_n]$. In [3] Bass proposed the following idea to attack the Jacobian Conjecture: the condition $\det JF \in \mathbb{C}^*$ implies that $\mathbb{C}[F]$ is a polynomial ring and that the derivations d/dF_i extend uniquely to n pairwise commuting derivations on $\mathbb{C}[X]$. In this way $\mathbb{C}[X]$ becomes a left module over the nth Weyl algebra $A_n(F) := \mathbb{C}[F, d/dF_1, \dots, d/dF_n]$. The derivations $\varepsilon_{ij} = F_i d/dF_j$ span $\mathfrak{gl}_n(\mathbb{C})$ in $A_n(F)$. It is shown in [3] that if $\mathfrak{g}$ is any Lie subalgebra of $\mathfrak{gl}_n(\mathbb{C})$ of dimension $> n$, then $\mathbb{C}[X]$ (and hence $\mathbb{C}[X]/\mathbb{C}[F]$) is a torsion module over the universal enveloping algebra $U(\mathfrak{g})$. The strategy proposed in [3] is to show that $\mathbb{C}[X]/\mathbb{C}[F]$ is a torsionfree $U(\mathfrak{g})$-module and hence is equal to zero. So, for example, to prove the Jacobian Conjecture for the case $n = 2$ it would be sufficient to prove that $\mathbb{C}[X]/\mathbb{C}[F]$ is a torsionfree module over $U(b) = \mathbb{C}[\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}]$. In fact, this case is studied extensively in [3] and several partial results are obtained; it is shown that $\mathbb{C}[X]/\mathbb{C}[F]$ is torsionfree over $\mathbb{C}[\varepsilon_{11} + \varepsilon_{22}, \varepsilon_{12}]$ over $\mathbb{C}[\partial]$ for all $\partial \in \mathfrak{gl}_2(\mathbb{C})$ and over $\mathbb{C}[\varepsilon_{11}, \varepsilon_{22}]$. In particular, the proof of the last result is spectacular and rather involved; apart from several algebraic tools it uses Siegel's theorem on algebraic curves with infinitely many integer points and Fabry's theorem on Gap series.

In this paper we introduce a new class of polynomial maps, the so-called nice polynomial maps. Using the notion of Eulerian operators (as introduced in [2])

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we show how for these polynomial maps the results obtained by Bass in [3] are rather easy to prove. The restriction that F is nice is not essential, since we show in §3 that every polynomial map satisfying the Jacobian Conjecture $\det JF \in k^*$ is "equivalent" with a nice polynomial map; i.e., we show that $F_{(\lambda)}(X) := F(X+\lambda) - F(\lambda)$ is nice for almost all $\lambda \in k^n$.

Let us finally sketch how to prove that $M := \mathbb{C}[X]/\mathbb{C}[F]$ has no $\mathbb{C}[\varepsilon_{11}, \varepsilon_{22}]$-torsion (the difficult case in [3]). So assume F is nice. We show first that for such an F we have a canonical inclusion $O(F_2)/\mathbb{C}[F_1] \hookrightarrow \mathbb{C}[[F_1]]/\mathbb{C}[F_1]$, where $O(F_2) = \mathbb{C}[X]/F_2\mathbb{C}[X]$. Suppose that M has $\mathbb{C}[\varepsilon_{11}, \varepsilon_{22}]$-torsion. Then it follows easily that M/F_2M has $\mathbb{C}[\varepsilon_{11}]$-torsion. Since $M/F_2M \simeq O(F_2)/\mathbb{C}[F_1]$, the above inclusion implies that $\mathbb{C}[[F_1]]/\mathbb{C}[F_1]$ has $\mathbb{C}[\varepsilon_{11}]$-torsion. This is a contradiction since all nonzero operators of $\mathbb{C}[\varepsilon_{11}]$ are Eulerian (an operator $P \in A_1 = \mathbb{C}[F_1, d/dF_1]$ is called Eulerian if $\mathbb{C}[[F_1]]/\mathbb{C}[F_1]$ has no P-torsion).

1. Eulerian operators

In this section we collect some facts concerning (linear) Eulerian operators (see also [2]). Let k be a field of characteristic zero, $k[X] := k[X_1, \dots, X_n]$ the polynomial ring in n variables over k, $\widehat{\mathscr{O}} := k[[X]] = k[[X_1, \dots, X_n]]$ the ring of formal power series over k, and $A_n := k[X][\partial_1, \dots, \partial_n]$ the nth Weyl algebra over k, i.e., the ring of differential operators over the polynomial ring $k[X]$. According to [2], an element $P \in A_n$ is called *Eulerian* if and only if for every polynomial $p \in k[X]$ every formal power series solution $g \in \widehat{\mathscr{O}}$ of the equation $Pg = p$ belongs to $k[X]$. In other words, the A_n-module $\widehat{\mathscr{O}}/k[X]$ has no P-torsion. From this definition we obtain immediately

(1.1) If P_1 and P_2 are Eulerian, so is P_1P_2.

(1.2) If P_1P_2 is Eulerian, so is P_2.

The Eulerian operators in case $n = 1$ are easy to describe.

Proposition 1.3 [1, Remarque 2.7]. *Let $Q \in A_1$. Then Q is Eulerian if and only if $Q = X_1^r P(X_1\partial_1)$ for some $r \in \mathbb{Z}$ and $0 \neq P(X_1\partial_1) \in k[X_1\partial_1]$.*

An example of an Eulerian operator in n variables is the Euler operator $\varepsilon := X_1\partial_1 + \cdots + X_n\partial_n$. More generally, every nonzero polynomial $P(\varepsilon) \in k[\varepsilon]$ is also Eulerian. In fact, this is a special case of the following result: let $P \in k[X_1, \dots, X_n]$ be a nonzero polynomial. To it we associate the differential operator $\widetilde{P} := P(X_1\partial_1, \dots, X_n\partial_n)$.

Proposition 1.4 [2, Proposition 2.3]. *$\widetilde{P}$ is Eulerian if and only if the equation $P(x) = 0$ has only a finite number of solutions in $\overline{\mathbb{N}}^n$.*

This result reveals a relationship between Eulerian operators and Diophantine geometry. It gives a large class of nontrivial Eulerian operators, namely, all operators $\widetilde{P}$ associated to curves with a finite number of integer solutions. In particular, the Fermat conjecture is equivalent with: $(X_1\partial_1+1)^n + (X_2\partial_2+1)^n - (X_3\partial_3+1)^n$ is Eulerian for all $n \geq 3$. From these examples it is evident that it is extremely difficult to describe Eulerian operators in the case $n \geq 2$. A more modest approach, therefore, is to study first Eulerian operators of small order (an operator $0 \neq P \in A_n$ is called of order $d \geq 0$ if $P = \sum_{|\alpha| \leq d} a_\alpha \partial^\alpha$, with $\sum_{|\alpha|=d} a_\alpha \partial^\alpha \neq 0$). As we will show below, pre-established understanding of Eulerian operators of order zero is very useful.

Now we will describe all Eulerian operators of order zero. Let $a \in k[X]$ be a nonconstant polynomial (a nonzero constant is obviously Eulerian). Write $a = p_1^{e_1} \cdots p_r^{e_r}$, the prime factor decomposition of a. So each p_i is irreducible in $k[X]$ and $e_i \geq 1$ for every i. From (1.1) and (1.2) it follows that a is Eulerian if and only if each p_i is Eulerian. So it remains to describe which irreducible polynomials in $k[X]$ are Eulerian.

Proposition 1.5. *Let $p \in k[X]$ be irreducible. Then p is Eulerian if and only if $p(0) = 0$.*

Proof. ($\rightarrow$) If $p(0) \neq 0$ then p is a unit in $\widetilde{\mathscr{O}}$. So p^{-1} exists in $\widehat{\mathscr{O}}$. Obviously $p^{-1} \notin k[X]$ (for otherwise $p \in k^*$). Then $p(p^{-1} + k[X]) = 0$ in $\widetilde{\mathscr{O}}/k[X]$, a contradiction since p is Eulerian.

($\leftarrow$) We need to show $p\widehat{\mathscr{O}} \cap k[X] = pk[X]$. Since $k[X]_{(X)} \subset \widehat{\mathscr{O}}$ is faithfully flat, we get $pk[X]_{(X)}\widehat{\mathscr{O}} \cap k[X]_{(X)} = pk[X]_{(X)}$, whence $p\widehat{\mathscr{O}} \cap k[X] = pk[X]_{(X)} \cap k[X]$. Since the prime ideal $pk[X]$ does not meet the multiplicatively closed set $k[X]\backslash(X)$ (for $p(0) = 0$), we have $pk[X]_{(X)} \cap k[X] = pk[X]$, as desired.

2. Some useful inclusions

In this section we consider the following situation: k is a field of characteristic zero and $F := (F_1, \dots, F_n)\colon k^n \to k^n$ is a polynomial map, i.e., each F_i belongs to $k[X]$. We assume that $\det JF \in k^*$ and $F(0) = 0$. The local inverse function theorem implies that $k[[X]] = k[[F]] := k[[F_1, \dots, F_n]]$, the formal power series ring in $F_1, \dots, F_n$ over k. Consequently,

$$k[[X]]/F_n k[[X]] \simeq k[[F_1, \dots, F_{n-1}]].$$

We identify these two rings and get a canonical map

$$k[X] \to k[[X]]/F_n k[[X]] = k[[F_1, \dots, F_{n-1}]];$$

its kernel equals $k[X] \cap F_n k[[X]]$. Now *assume* that F_n is irreducible in $k[X]$. Then Proposition 1.5 implies that we get the inclusion

$$\mathscr{O}(F_n) := k[X]/F_n k[X] \hookrightarrow k[[F_1, \dots, F_{n-1}]]. \tag{2.1}$$

Furthermore, we have the canonical map $k[F_1, \dots, F_{n-1}] \to \mathscr{O}(F_n)$ with kernel $k[F_1, \dots, F_{n-1}] \cap F_n k[X]$. This kernel is zero since it is contained in $k[F_1, \dots, F_{n-1}] \cap F_n k[[F]] = (0)$. Hence, we get the inclusion

$$k[F_1, \dots, F_{n-1}] \hookrightarrow \mathscr{O}(F_n). \tag{2.2}$$

Summarizing, we get

Proposition 2.3. *Let $F\colon k^n \to k^n$ be a polynomial map such that $\det JF \in k^*$, F_n irreducible in $k[X]$, and $F(0) = 0$. Then we have inclusions (2.1) and (2.2). In particular, $\mathscr{O}(F_n)/k[F_1, \dots, F_{n-1}] \hookrightarrow k[[F_1, \dots, F_{n-1}]]/k[F_1, \dots, F_{n-1}]$.*

3. Nice polynomial maps

Let k be an algebraically closed field of characteristic zero and $F\colon k^n \to k^n$ a polynomial map. Put $M := k[X]/k[F]$. So M is a $k[F]$-module. Let $1 \leq p \leq n$. We call F *p-nice* if F_p is irreducible in $k[X]$, $\bigcap_{i \geq 1} F_p^i M = (0)$, and $\det JF \in k^*$. If F is p-nice for all $1 \leq p \leq n$, we call F *nice.* For each $\lambda \in k^n$ we define the polynomial map $F_{(\lambda)}$ by $F_{(\lambda)}(X) := F(X + \lambda) - F(\lambda)$. So $F_{(\lambda)}(0) = 0$.

Proposition 3.1. *Let* $\det JF \in k^*$. *Then* $F_{(\lambda)}$ *is nice for almost all* $\lambda \in k^n$ *(i.e., for all* λ *in a Zariski open set of* k^n*).*

Proof. By Lemma 3.2 and Corollary 3.4 it follows that $F - \mu$ is nice for almost all $\mu \in k^n$, i.e., for all μ outside some hypersurface $f^{-1}(0)$. So $F - F(\lambda)$ is nice for all λ outside $(f \circ F)^{-1}(0)$. Since ϕ defined by $\phi(X) = X + \lambda$ is a polynomial automorphism of $k[X]$, the composition $(F - F(\lambda)) \circ \phi$ is also nice for almost all $\lambda \in k^n$, which completes the proof.

Lemma 3.2. *Let* $f \in k[X_1, \dots, X_n]$ *be such that* $1 \in (\partial f/\partial X_1, \dots, \partial f/\partial X_n)$. *Then there exists a finite subset of* E *of* k *such that* $f - \lambda$ *is irreducible in* $k[X]$ *for all* $\lambda \in k \backslash E$.

Proof. Consider $f(X) - Z \in k[Z, X]$. This polynomial is irreducible in $k[Z][X]$ and hence in $k(Z)[X]$ by Gauss's lemma. Furthermore $\deg_Z f(X) - Z = 1$, so by Bertini's theorem (see [5, §11, Theorem 18] and observe that in the proof given there the hypothesis "$F(x, \lambda^*)$ is irreducible in k for every λ^*" can be replaced by: for infinitely many λ^* in k), we obtain that if $f(X) - \lambda$ is reducible for infinitely many $\lambda \in k$, then there exist polynomials ϕ, $\chi \in k[X]$, $m \in \mathbb{Z}$, and $a_i(Z) \in k[Z]$ such that

$$f(X) - Z = a_0(Z)\phi^m + a_1(Z)\phi^{m-1}\chi + \cdots + a_m(Z)\chi^m \tag{3.1}$$

and

$$\deg_X f > \max(\deg_X \phi, \deg_X \chi). \tag{3.2}$$

Consequently, $m \geq 2$, for if $m = 1$ then $f(X) - Z = a_0(Z)\phi + a_1(Z)\chi$ and then $\deg_X f \leq \max(\deg_X \phi, \deg_X \chi)$, which contradicts (2). Comparing the coefficients of Z^0 and Z^1 in (1), we get

$$f(X) = \sum_{i=0}^{m} a_i(0)\phi^{m-i}\chi^i \tag{3.3}$$

and

$$-1 = \sum_{i=0}^{m} a_i'(0)\phi^{m-i}\chi^i = H(\phi, \chi) \tag{3.4}$$

where $H(U_1, U_2) = \sum a_i'(0)U_1^{m-i}U_2^i$ is a homogeneous polynomial in U_1, U_2. Write $H = \prod_{i=1}^{m}(\alpha_i U_1 + \beta_i U_2)$ with α_i, $\beta_i \in \overline{k}$ (an algebraic closure of k). Then $H(\phi, \chi) = -1$ implies that $\alpha_i \phi + \beta_i \chi \in \overline{k}^*$ for all i. So if, for example $\alpha_1 \neq 0$ then $\phi = \beta\chi + \lambda$ for some $\beta, \lambda \in \overline{k}$. Then by (3) $f(X) = \sum_{i=0}^{m} a_i(0)(\beta\chi + \lambda)^{m-i}\chi^i = g(\chi)$ for some $g(T) \in \overline{k}[T]$. Now observe that $\deg_T g(T) \geq 2$, for if $\deg_T g(T) \leq 1$, say $g(T) = \mu_1 T + \mu_0$, then $f(X) = \mu_1\chi + \mu_0$, so $\deg_X f(X) \leq \deg_X \chi$, which contradicts (2). So $f(X) = g(\chi)$ with $g(T) \in \overline{k}[T]$, $\deg g(T) \geq 2$. But then $\partial f/\partial X_i = g'(\chi)\partial\chi/\partial X_i$. Since $\deg g'(T) \geq 1$, $g'(z) = 0$ for some $z \in \overline{k}$. Then take $x \in \overline{k}^n$ with $\chi(x) = z$. We get $g'(\chi(x)) = 0$, hence $\partial f(x)/\partial X_i = 0$ for all i, which is a contradiction with $1 \in (\partial f/\partial X_1, \dots, \partial f/\partial X_n)$. So the hypothesis, $f(X) - \lambda$ is reducible for infinitely many $\lambda \in k$, leads to a contradiction, which proves the lemma.

Lemma 3.3. *Let* $\det JF \in k^*$. *There exists* $0 \neq f \in k[F]$ *such that the following holds: if* I *is an ideal in* $k[F]$ *with* $\bigcap_{p \geq 1} I^p M \neq (0)$, *then* $f \in r(I)$.

Proof. Let $d = |k(X): k(F)|$. By the primitive element theorem there exists an element g in $k(X)$, which we can assume to be integral over $k[F]$, such that $k(X) = k(F)[g]$. So there exists $0 \neq f \in k[F]$ such that $f \cdot X_i \in k[F][g]$ for all i. Since $g^d \in \sum_{i=0}^{d-1} k[F]g^i$ (g being integral over $k[F]$), it follows that $k[X] \subset \bigoplus_{i=1}^{d-1} k[F]_f g_i$, and since M has no $k[F]$ torsion (because $k[X] \cap k(F) = k[F]$ [3, Corollary 1.3]), we conclude that $M \subset M_f \subset \bigoplus_{i=1}^{d-1} k[F]_f g_i$. Now assume that $\bigcap_{p \geq 1} I^p M \neq (0)$. It follows that $\bigcap I^p(\bigoplus_{i=1}^{d-1} k[F]_f g^i) \neq 0$ and hence that $\bigcap I^p k[F]_f \neq (0)$. By Krull's intersection theorem we conclude that $Ik[F]_f = k[F]_f$, which implies that $f \in r(I)$.

Corollary 3.4. *Let* $\det JF \in k^*$. *Then* $I(\lambda) := \bigcap_{i=1}^{\infty} (F_n - \lambda)^i M = (0)$ *for all* $\lambda \in k$ *outside some finite subset of* k.

Proof. Let f be as in Lemma 3.3. Let $\lambda \in k$ be such that $I(\lambda) \neq (0)$ and $F_n - \lambda$ irreducible. Then Lemma 3.3 implies that $F_n - \lambda$ divides f. Since f has only a finite number of irreducible factors, and since $F_n - \lambda$ is irreducible for all $\lambda \in k$ outside a finite subset of k [by Lemma 3.2, since $1 \in (\partial F_n/\partial X_1, \ldots, \partial F_n/\partial X_n)$ because $\det JF \in k^*$], the corollary follows.

4. Applications to the case $n = 2$

Let again $F: k^n \to k^n$ be a polynomial map with $\det JF \in k^*$ and k an algebraically closed field of characteristic zero. First recall (see [3] or [4]) that the Jacobian condition $\det JF \in k^*$ implies that the derivations d/dF_i on $k[F]$ can be extended uniquely to pairwise commuting derivations on $k[X]$. In other words, $k[X]$ becomes a left ($A_n(F) = k[F, d/dF_1, \ldots, d/dF_n]$)-module. The derivations $\varepsilon_{ij} := F_i d/dF_j$ span $\mathfrak{gl}_n(k) \subset A_n(F)$. The strategy proposed in [3] is to attack the Jacobian Conjecture by showing that $M := k[X]/k[F]$ is a torsionfree module over the enveloping algebra $U(\mathfrak{g})$ for some Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}_n(k)$ of dimension $> n$.

Observe that if $\lambda \in k^n$, then F is invertible if and only if the polynomial morphism $F_{(\lambda)} := F(X + \lambda) - F(\lambda)$ is invertible. So in order to prove the Jacobian Conjecture we may replace F by $F_{(\lambda)}$, and hence by Proposition 3.1 we assume from now on: F *is nice and* $F(0) = 0$. Now we will show how under these conditions the result obtained by Bass in §6 of [3] easily follows from the inclusion in Proposition 2.3 and the fact that each operator of $k[F_1 d/dF_1] \backslash \{0\}$ is Eulerian.

So let $n = 2$. To simplify the notations we put $x = F_1$, $y = F_2$, $X = X_1$, $Y = X_2$, $\varepsilon_x = x\partial_x$, $\varepsilon_y = y\partial y$. First observe that from Proposition 2.3 and the assumption that F is nice and $F(0) = 0$, it follows that $\bigcap x^p M = \bigcap y^p M = (0)$, $M/yM \subset k[[x]]/k[x]$, and $M/xM \subset k[[y]]/k[y]$.

Now we are able to prove the main result of this paper.

Theorem 4.1. *Let* $P = \sum_{i \geq 0} (y\partial_x)^i P_i(\varepsilon_x, \varepsilon_y) \in A_2$ *with* $P_0(\varepsilon_x, r) \neq 0$ *for all* $r \in \overline{\mathbb{N}}$. *If* $D \in A_2$ *is such that* M *has no D-torsion, then* M *has no DP-torsion.*

Proof. It suffices to show that M has no P-torsion. So let $0 \neq m \in M$ with $Pm = 0$. Since $\bigcap y^p M = (0)$, there exists $r \in \mathbb{N}$ with $m = y^r \tilde{m}$ with $\tilde{m} \notin yM$. Observe that $Py^r = y^r \widetilde{P}$ in A_2, where $\widetilde{P} = \sum (y\partial_x)^i P_i(\varepsilon_x, \varepsilon_y + r)$. Since

M has no y-torsion, it follows that $\widetilde{P}\tilde{m} = 0$. Observe that $\widetilde{P} = P_0(\varepsilon_x, r) + yQ$ for some $Q \in A_2$. So $P_0(\varepsilon_x, r)\overline{m} = 0$ in M/yM, where $\overline{m} = \tilde{m} + yM$, which is nonzero since $\tilde{m} \notin yM$. Since $M/yM \subset k[[x]]/k[x]$ and $P_0(\varepsilon_x, r)$ is Eulerian by Proposition 1.3, it follows that $\overline{m} = 0$, which is a contradiction.

Corollary 4.2. *M has no torsion over $k[\varepsilon_x, \varepsilon_y]$, $k[a\varepsilon_x + b\varepsilon_y, y\partial_x]$ $(a, b \in k, a \neq 0)$, and $k[\partial]$ where $\partial \in \mathfrak{gl}_2(k)$.*

Proof. (i) From Theorem 4.1 it follows in particular that M has no ∂_x torsion and no $\partial_x - \lambda$ torsion for all $\lambda \in k$. Interchanging the roles of x and y the same holds for ∂_y and $\varepsilon_y - \lambda$.

(ii) Let $0 \neq P(\varepsilon_x, \varepsilon_y) \in k[\varepsilon_x, \varepsilon_y]$ and assume $Pm = 0$. Dividing the polynomial $P(X, Y)$ by possible factors of the form $(Y - r)^e$, $e \geq 1$, $r \in \overline{\mathbb{N}}$, we can write $P(X, Y) = \prod_i (Y - r_i)^{e_i} \widetilde{P}(X, Y)$ with $\widetilde{P}(X, r) \neq 0$ for all $r \in \overline{\mathbb{N}}$. Then by (i) $Pm = 0$ implies $\widetilde{P}(\varepsilon_x, \varepsilon_y)m = 0$. Then apply Theorem 4.1, which gives a contradiction, proving the first case.

(iii) Since M has no $y\partial_x$-torsion (by (i)), the second case follows readily from Theorem 4.1, observing that $P_0(a\varepsilon_x + br) \neq 0$ for all $r \in \overline{\mathbb{N}}$ if $P_0(aX + bY) \neq 0$.

(iv) After a linear change of coordinates in $k[x, y]$ we can assume that ∂ is in Jordan canonical form. Then either $\partial = a(\varepsilon_x + \varepsilon_y) + y\partial_x$ or $\partial = a\varepsilon_x + b\varepsilon_y$, $a, b \in k$. Then use the second case proved before. □

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