

WEAK COMPACTNESS IN $L^1(\mu, X)$

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ABSTRACT. We present a characterization of weak compactness in $L^1(\mu, X)$ and in more general Banach spaces of vector-valued measurable functions. Moreover, we slightly refine Talagrand's parametrized version of Rosenthal's l^1 -theorem and extend it to $L^1(\mu, X)$ -bounded sequences.

1. INTRODUCTION

This note addresses the problem of characterizing weak compactness in the space $L^1(\mu, X)$ of (equivalence classes of) Bochner integrable functions from a probability space (Ω, Σ, μ) into a Banach space X , endowed with its usual norm $\|f\|_1 = \int \|f\| d\mu$ (cf. [1, 2] and the discussion in [3, Chapter IV.2]). Recently, Ülger [12, Theorem 4] presented a characterization of the weakly relatively compact subsets A of $L^1(\mu, X)$ that are L^∞ -bounded as those for which, given any sequence $(f_n)_n \subset A$, there exists a sequence $(\tilde{f}_n)_n$ with $\tilde{f}_n \in \text{co}\{f_k \mid k \geq n\}$ such that $(\tilde{f}_n(\omega))_n$ is weakly convergent in X for a.e. $\omega \in \Omega$, X a general Banach space. Ülger's proof relies in an essential manner on the deep analysis by Talagrand [11] of the parametrized version for $L^\infty(\mu, X)$ of Rosenthal's l^1 -theorem as well as on James's characterization of weak compactness.

In §2 of this note we remove the restriction of L^∞ -boundedness and give a short and completely elementary proof of the analogous characterization of general weakly compact subsets of $L^1(\mu, X)$ solely based on the classical weak compactness criteria in Banach spaces (Theorem 2.1). Using this characterization, we slightly refine the above-mentioned result by Talagrand [11, Theorem 2.1] and also extend it to the case of $L^1(\mu, X)$ -bounded sequences (Theorem 2.4). Extensions of the weak compactness criterion to more general Banach spaces of vector-valued measurable functions are presented in §3.

2. WEAK COMPACTNESS IN $L^1(\mu, X)$

We first place Ülger's criterion—extended to general weakly relatively compact subsets of $L^1(\mu, X)$ —into the context of the classical equivalences to weak compactness:

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Eberlein, Smul'yan, Grothendieck (cf. [9, §24, 3.(8); 7]): For a subset A of a Banach space X , the following are equivalent to weak relative compactness: (a) weak relative sequential compactness; (b) weak relative convex compactness; (c) boundedness and the interchangeable-double-limits-property.

Theorem 2.1. *Let A be a bounded subset of $L^1(\mu, X)$. Then the following are equivalent:*

- (1) A is weakly relatively compact.
- (2) A is uniformly integrable, and, given any sequence $(f_n)_n \subset A$, there exists a sequence $(g_n)_n$ with $g_n \in \text{co}\{f_k \mid k \geq n\}$ such that $(g_n(\omega))_n$ is norm convergent in X for a.e. $\omega \in \Omega$.
- (3) A is uniformly integrable, and, given any sequence $(f_n)_n \subset A$, there is a sequence $(g_n)_n$ with $g_n \in \text{co}\{f_k \mid k \geq n\}$ such that $(g_n(\omega))_n$ is weakly convergent in X for a.e. $\omega \in \Omega$.

Proof. Suppose (1) holds. Then A is uniformly integrable (cf. [3]), and if $(f_n)_n$ is any sequence in A then some subsequence $(f_{n_k})_k$ converges weakly to some $f \in L^1(\mu, X)$, and thus there exists a sequence $\tilde{g}_k \in \text{co}\{f_{n_m} \mid m \geq k\}$, $k \in \mathbb{N}$, such that $\|\tilde{g}_k - f\|_1 \rightarrow 0$. Hence, a subsequence $(g_n)_n$ of $(\tilde{g}_k)_k$ converges to f pointwise a.e. in the norm of X , thus establishing (2).

(2) implying (3) being obvious, it remains to prove that (3) implies (1). Referring to (c) in the Eberlein-Smul'yan-Grothendieck Theorem (hereafter referred to as the E-S-G-Theorem), we have to show that, given sequences $(f_m)_m \subset A$ and $(\varphi_n)_n \subset \{\varphi \in L^1(\mu, X)^* \mid \|\varphi\| \leq 1\}$, we have

$$\alpha := \lim_n \lim_m \varphi_n(f_m) = \lim_m \lim_n \varphi_n(f_m) =: \beta,$$

provided the iterated limits exist [7, Corollaire 1 of Théorème 7].

Given such sequences $(f_m)_m$ and $(\varphi_n)_n$, let $(g_m)_m$ be the sequence associated with $(f_m)_m$ according to (3), with $E \in \Sigma$, $\mu(E) = 0$, the exceptional subset of Ω . Define $g : \Omega \rightarrow X$ by $g(\omega) = \text{weak-limit } g_n(\omega)$ for $\omega \in \Omega \setminus E$, and by $0 \in X$, otherwise. Clearly, g is essentially separably valued and weakly measurable, hence strongly measurable. Moreover, by its very definition, and using Fatou's Lemma and boundedness of A ,

$$\int \|g\| d\mu \leq \int \liminf \|g_n\| d\mu \leq \liminf \int \|g_n\| d\mu < \infty,$$

so that $g \in L^1(\mu, X)$. Let us show that the sequence $(g_n)_n$ converges weakly in $L^1(\mu, X)$ to g . First note that we can assume $L^1(\mu)$ to be separable: consider a countably generated sub-sigma algebra Σ_1 of Σ such that $(g_m)_m \subset L^1(\mu, \Sigma_1, X)$. Then, according to [5, Chapter VI.8, Corollary 7], the continuous linear functionals \tilde{h} on $L^1(\mu, X)$ are represented by w^* -measurable functions $h : \Omega \rightarrow X^*$ such that $\|h(\cdot)\| \in L^\infty(\mu)$, the pairing being given by $\langle \tilde{h}, f \rangle = \int \langle h, f \rangle d\mu$, $f \in L^1(\mu, X)$. Given any such h , uniform integrability of the sequence $(\langle h, g_n \rangle)_n$ and the fact that $\langle h(\cdot), g_n(\cdot) \rangle \rightarrow \langle h(\cdot), g(\cdot) \rangle$ a.e. Ω , in conjunction with Vitali's convergence theorem imply that $\langle \tilde{h}, g_n \rangle \rightarrow \langle \tilde{h}, g \rangle$, thus proving our assertion.

To complete the proof, note that if $h_m = \sum_1^r \alpha_i f_{k_i}$ is any convex combination of the $(f_m)_m$, and $\varphi \in L^1(\mu, X)^*$ is such that $\gamma := \lim_m \varphi(f_m)$ exists, then

$$|\gamma - \varphi(h_m)| = \left| \sum_1^r \alpha_i (\gamma - \varphi(f_{k_i})) \right| \leq \max\{|\gamma - \varphi(f_{k_i})| \mid 1 \leq i \leq r\}.$$

This shows that $\alpha = \varphi(g) = \beta$, where φ is a weak*-cluster point of $(\varphi_n)_n$, thus completing the proof. \square

An obvious special case of Theorem 2.1 (with Ω consisting of just one element) is the following version of equivalence (b) in the E-S-G-Theorem.

Corollary 2.2. *For a subset A of a Banach space X the following are equivalent:*

- (1) *A is weakly relatively compact.*
- (2) *Given a sequence $(x_n)_n \subset A$, there exists a sequence $(y_n)_n$ with $y_n \in \text{co}\{x_k \mid k \geq n\}$ that is norm convergent.*
- (3) *Given a sequence $(x_n)_n \subset A$, there exists a sequence $(y_n)_n$ with $y_n \in \text{co}\{x_k \mid k \geq n\}$ that is weakly convergent.*

Remarks 2.3. 1. The equivalence of propositions (1) and (3) of Corollary 2.2 is the content of [12, Lemma 2.1], which Ülger proved by using James's characterization of weak compactness via norm-attaining functionals.

2. We deduced Corollary 2.2 as a special case of its parametrized version given in Theorem 2.1. One can turn around and, conversely, deduce Theorem 2.1 from Corollary 2.2: note that, in the course of our proof of Theorem 2.1, we actually showed that proposition (3) of this result implies proposition (3) of Corollary 2.2 for the Banach space $L^1(\mu, X)$. In this sense, Theorem 2.1 and the E-S-G-Theorem are equivalent.

3. While Theorem 2.1 is the parametrized version of part (b) of the E-S-G-Theorem we note in passing that the analogous parametrization of part (a)—sequences out of a weakly relatively compact subset of $L^1(\mu, X)$ allowing for subsequences converging weakly a.e.—does not hold: the sequence $(r_n)_n$ of the Rademacher functions in $L^1(0, 1)$ puts that to rest even for $X = \text{reals}$.

A different example to this effect that, at the same time, shows to what extent the pointwise ranges of weakly relatively compact subsets of $L^1(\mu, X)$ can be pathological is the following one for $X = c_0$: Given any bounded sequence $(x_n)_n \in c_0$, let $K = \{r_n x_n : n \in \mathbb{N}\} \subseteq L^1([0, 1], c_0)$, where $(r_n)_n$ denotes the sequence of the Rademacher functions. Using the material of [3, IV.1, pp. 97–98], it is not hard to see that $(r_n x_n)$ converges to zero weakly in $L^1([0, 1], c_0)$. However, if $(x_n)_n$ is such that no subsequence converges weakly to zero in c_0 , then, for every subsequence of $(r_n x_n)_n$ and every $t \in [0, 1]$, the sequence $(r_{n_k}(t)x_{n_k})_k$ does not converge weakly in c_0 .

We now use Theorem 2.1 to present a slight refinement of Talagrand's parametrized version of Rosenthal's l_1 -theorem [11, Theorem 2.1] and, at the same time, extend it from $L^\infty(\mu, X)$ - to $L^1(\mu, X)$ -bounded sequences.

Theorem 2.4. *Assume that $(f_n)_n$ is a bounded sequence in $L^1(\mu, X)$. Then there exist $g_n \in \text{co}\{f_k \mid k \geq n\}$, $n \in \mathbb{N}$, and three measurable subsets C_1, C_2 , and L of Ω with $\mu(C_1 \cup C_2 \cup L) = 1$ such that*

- (a) *for $\omega \in C_1$, the sequence $(g_n(\omega))_n$ is norm-convergent in X ;*

(b) for $\omega \in C_2$, the sequence $(g_n(\omega))_n$ is weakly Cauchy but not weakly convergent in X ;

(c) for $\omega \in L$, there is $k \in \mathbb{N}$ such that the sequence $(g_n(\omega))_{n \geq k}$ is equivalent to the unit vector basis of l_1 .

Proof. Step 1. We first consider the special case where $(f_n)_n$ is a bounded sequence in $L^\infty(\mu, X)$. Then, by Talagrand's result [11, Theorem 2.1], there exist $h_n \in \text{co}\{f_k \mid k \geq n\}$, $n \in \mathbb{N}$, and measurable subsets C, L of Ω , $\mu(C \cup L) = 1$, such that for $\omega \in L$ (c) holds for $(h_n)_n$, while for $\omega \in C$ the sequence $(h_n(\omega))_n$ is weakly Cauchy.

Define C_2 as the subset of C consisting of those $\omega \in C$ where $(h_n(\omega))_n$ does not converge weakly in X . Note that C_2 is measurable. Indeed, there is no loss of generality in assuming that X is separable and has a Schauder basis. Denote by $P_{k,l}$ the canonical projection onto the space spanned by the basis elements $\{e_k, \dots, e_l\}$. We may write $C \setminus C_2$ as the subset of C consisting of those ω such that for $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that for all $l > k$ we have

$$\limsup_{n, m \rightarrow \infty} \|P_{k,l}(h_n(\omega) - h_m(\omega))\| < \varepsilon.$$

Clearly, this defines a measurable set.

Let $C_3 = C \setminus C_2$ and note that $(h_n \cdot \chi_{C_3})_n$ is a bounded sequence in $L^\infty(\mu, X)$ converging weakly pointwise. In particular, by Theorem 2.1, this sequence is weakly relatively compact in $L^1(\mu, X)$ and there are $g_n \in \text{co}\{h_k \mid k \geq n\}$ (the sets of indices of the convex combinations of the h_k 's even being disjointly supported) such that, for almost all $\omega \in C_3$, $(g_n(\omega))_n$ is norm-convergent in X . Defining C_1 to be this subset of C_3 , we have finished this part of the proof.

Step 2. The general case will be reduced to the above special case by means of the following observation: \square

Lemma 2.5. *If $(f_n)_n$ is a bounded sequence in $L^1(\mu, X)$, then there exist $g_n \in \text{co}\{f_k \mid k \geq n\}$, $n \in \mathbb{N}$, and a weight function $w \in L^0_+(\mu)$ such that $\sup\{\|g_n(\omega)\|_X \mid n \in \mathbb{N}\} \leq w(\omega)$ for all $\omega \in \Omega$.*

In fact, it now suffices to apply Step 1 to the sequence $(g_n/w)_n$. In order to prove Lemma 2.5, we consider the sequence $(v_n)_n \subset L^1(\mu)$ defined by $v_n(\omega) = \|f_n(\omega)\|_X$. By Komlos's theorem [8], there is a subsequence $(v_{n_k})_k$ that converges in Cesaro mean almost surely. It follows easily that we have disjointly supported convex combinations

$$u_n = \sum_{j=N_n+1}^{N_{n+1}} \lambda_j^{(n)} v_j \in \text{co}\{v_k \mid k \geq N_n + 1\}, \quad \lambda_j^{(n)} \geq 0, \quad \sum_{j=N_n+1}^{N_{n+1}} \lambda_j^{(n)} = 1,$$

such that $(u_n)_n$ converges almost surely. In particular, $w(\omega) := \sup_{n \in \mathbb{N}} u_n(\omega)$ is almost surely finite. Letting

$$g_n := \sum_{j=N_n+1}^{N_{n+1}} \lambda_j^{(n)} f_j,$$

we may estimate

$$\sup_{n \in \mathbb{N}} \|g_n(\omega)\|_X \leq \sup_{n \in \mathbb{N}} u_n(\omega) \leq w(\omega).$$

This completes the proof of Theorem 2.4.

A particular consequence of Theorems 2.1 and 2.4 is the following sufficient condition for weak relative compactness in $L^1(\mu, X)$.

Corollary 2.6 (Compare [2, Theorem 2; 12, Corollary 9]). *Assume that A is a bounded and uniformly integrable subset of $L^1(\mu, X)$ such that, for $f \in A$, one has $f(\omega) \in B_\omega$ a.e. $\omega \in \Omega$, where, for $\omega \in \Omega$, $B_\omega \subset X$ is weakly relatively compact. Then A is weakly relatively compact in $L^1(\mu, X)$.*

Finally, we address the following question: What happens if one replaces the convex combinations by subsequences in the statement of Theorem 2.1? Then (2) becomes a characterization of norm relatively compact subsets of $L^1(\mu, X)$. This is just a consequence of Vitali's convergence theorem, and the proof is left to the interested reader.

Proposition 2.7. *For a subset A of $L^1(\mu, X)$, the following are equivalent:*

- (1) *A is relatively norm-compact.*
- (2) *A is bounded and uniformly integrable, and for every sequence $(f_n)_n$ in A there is a subsequence $(f_{n_k})_k$ that converges almost surely with respect to the norm of X .*

3. EXTENSIONS AND RELATED RESULTS

Analyzing the proof of Theorem 2.1 reveals that we have used little information pertaining to the special nature of $L^1(\mu)$. Indeed, let E be an order continuous Banach lattice with a weak unit. By a well-known representation theorem [10, Theorem 1.b.14], we may assume that $L^\infty(\Omega, \Sigma, \mu) \subseteq E \subseteq L^1(\Omega, \Sigma, \mu)$ for some probability space (Ω, Σ, μ) , E being an ideal in $L^1(\mu)$, the inclusion maps being continuous; moreover, we may identify E^* with the space of (equivalence classes of) measurable functions g on (Ω, Σ, μ) such that

$$\|g\|_{E^*} = \sup \left\{ \int f g \, d\mu \mid \|f\|_E \leq 1 \right\} < \infty.$$

The characterization of weakly compact subsets of E has been known for a long time (cf. [4; 6, Theorem 83I]):

Proposition 3.1. *Let E be an order continuous Banach lattice with weak unit represented as above with $L^\infty(\mu) \subseteq E \subseteq L^1(\mu)$. A subset $K \subseteq E$ is weakly relatively compact if and only if, for every $g \in E^*$, the set $\{fg \mid f \in K\}$ is uniformly integrable.*

We may now define the space $E(X)$ of strongly measurable functions $f : \Omega \rightarrow X$ such that $\|f(\cdot)\|_X \in E$, equipped with its usual norm (compare [11, §3]). Let us formulate the abstract version of Theorem 2.1 for the space $E(X)$.

Theorem 3.2. *Let E be an order continuous Banach lattice with weak unit, represented as above with $L^\infty(\mu) \subseteq E \subseteq L^1(\mu)$. Then, for a subset A of $E(X)$, the following are equivalent:*

- (1) A is weakly relatively compact.
- (2) The subset $\{\|f(\cdot)\|_X \mid f \in A\}$ of E is weakly relatively compact, and, given any sequence $(f_n)_n$ in A , there exists a sequence $(g_n)_n$ with $g_n \in \text{co}\{f_k \mid k \geq n\}$, and such that $(g_n(\omega))_n$ is norm convergent for a.e. $\omega \in \Omega$.
- (3) As in (2) except for $(g_n(\omega))_n$ being weakly convergent for a.e. $\omega \in \Omega$.

Proof. The proof is completely analogous to the proof of Theorem 2.1. Suppose that (1) holds. According to Proposition 3.1 above, in order to prove that (1) implies the first assertion of (2), it is enough to show that for every $g \in E^*$ the set $\{g(\cdot)\|f(\cdot)\|_X \mid f \in A\}$ is uniformly integrable. Noting that the set $\{g \cdot f \mid f \in A\}$ is weakly relatively compact in $L^1(\mu, X)$, this follows from (the easy part of) Theorem 2.1. The second assertion of (2) is immediate from Theorem 2.1

(2) implying (3) being obvious, it remains to show that (3) implies (1). Here, we use Corollary 2.2: given a sequence $(f_n)_n \subset A$, we have to find $g_n \in \text{co}\{f_k \mid k \geq n\}$, $n \in \mathbb{N}$, such that $(g_n)_n$ is weakly convergent in $E(X)$. By hypothesis, there exist $g_n \in \text{co}\{f_k \mid k \geq n\}$, $n \in \mathbb{N}$, such that $(g_n(\omega))_n$ is weakly convergent in X for a.e. $\omega \in \Omega$.

The function $g(\omega) = \text{weak-lim } g_n(\omega)$ a.e. $\omega \in \Omega$, which, as in the proof of Theorem 2.1, is strongly measurable, is an element of $E(X)$: According to [4; III.10; 10, 1.b], we have to show that $f \cdot \|g(\cdot)\|_X \in L^1(\mu)$ for all $f \in E^*$, $f \geq 0$. As in the proof of Theorem 2.1, this is a consequence of Fatou's Lemma.

Finally, note that any $\tilde{h} \in E(X)^*$ can be represented by a weak-star-measurable function $h : \Omega \rightarrow X^*$ such that $\|h(\cdot)\| \in E^*$, the pairing being given by

$$\langle \tilde{h}, f \rangle = \int \langle h(\omega), f(\omega) \rangle d\mu(\omega), \quad f \in E(X).$$

As, by hypothesis, the sequence $(\langle h(\cdot), g_n(\cdot) \rangle)$ is uniformly integrable, the fact that $\langle \tilde{h}, g_n \rangle \rightarrow \langle \tilde{h}, g \rangle$ for all $\tilde{h} \in E(X)^*$ once again follows from Vitali's convergence theorem just as in the proof of Theorem 2.1. This shows that $(g_n)_n$ converges to g weakly in $E(X)$, and thus completes the proof of Theorem 3.2.

Corollary 3.3. *Under the assumptions of Theorem 3.2, conditions (1)–(3) are also equivalent to:*

- (4) The subset $\{\|f(\cdot)\|_X \mid f \in A\}$ of E is weakly relatively compact in E , and A is weakly relatively compact in $L^1(\mu, X)$.

In closing this paper, we note that Theorem 3.2 reduces to a particularly easy criterion when applied to reflexive lattices E . As an example, we single out the case of $E = L^p(\mu)$, $1 < p < \infty$.

Corollary 3.4. *Suppose $1 < p < \infty$, and let A be a bounded subset of $L^p(\mu, X)$. Then the following are equivalent:*

- (1) A is weakly relatively compact.
- (2) For each sequence $(f_n)_n$ in A , there exists a sequence $(g_n)_n$ with $g_n \in \text{co}\{f_k \mid k \geq n\}$ such that $(g_n(\omega))_n$ is norm convergent for a.e. $\omega \in \Omega$.
- (3) For each sequence $(f_n)_n$ in A , there exists a sequence $(g_n)_n$ with $g_n \in \text{co}\{f_k \mid k \geq n\}$ such that $(g_n(\omega))_n$ is weakly convergent for a.e. $\omega \in \Omega$.

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