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AN EXAMPLE OF A CARATHÉODORY COMPLETE BUT NOT FINITELY COMPACT ANALYTIC SPACE

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ABSTRACT. An analytic space is given which is c_X-complete but not c_X-finitely compact.

1. INTRODUCTION

If X is a (connected) complex analytic space, the Carathéodory pseudodistance c_X on X is given by

$$c_X(x, y) = \sup\{\omega(f(x), f(y)) : f \in \mathscr{O}(X, \Delta)\},$$

where $\mathscr{O}(X, \Delta)$ denotes the set of all holomorphic mappings from X to the open unit disc $\Delta \subset \mathbb{C}$ and where ω is the hyperbolic (Poincaré) distance on Δ. Now let us consider an analytic space X for which c_X is a distance defining the topology of X (cf. [2, 7]). If the c_X-balls are relatively compact in X, i.e., X is finitely compact with respect to c_X, then it is clear that X is c_X-complete. The problem of the equivalence of these two notions has been raised by Kobayashi [4] (see also [1]). A positive answer for plane domains was given by Selby [5] and Sibony [6]. (We thank C. J. Earle for showing us the paper of Selby.) The purpose of this note is to construct an analytic space X, c_X-complete but not finitely compact with respect to c_X. The case of domains in $\mathbb{C}^n$, $n > 1$, still remains open.

2. CONSTRUCTION OF THE EXAMPLE

For every integer $n > 0$ let

$$p_k^n := \left(1 - \frac{1}{n+1}\right) \exp\left(\frac{2\pi i k}{n+1}\right) \in \Delta, \qquad 0 \le k \le n.$$

We construct a connected analytic space X in the following way:

Let $(D_j)_{j=0}^{\infty}$ be a sequence of copies of Δ, i.e., $D_j := \Delta$, and let $q_k^n \in D_0$ be defined by $q_k^n := p_k^n$, $n > 0$, $0 \le k \le n$. Moreover, for every $n > 0$, let $x_k^n \in D_n$ be given by $x_k^n := p_k^n$, $0 \le k \le n$.

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Then the analytic space X is obtained by patching together D_n and D_0, $n > 0$, by identifying the points q_k^n and x_k^n, $0 \leq k \leq n$.

Topologically, X is the quotient of $\bigcup_{n=0}^{\infty} D_n$ by the equivalence relation $\mathscr{R}: q_k^n \mathscr{R} x_k^n$, $n > 0$, $k = 0, \ldots, n$. The analytic structure of X is defined by gluing D_0 and D_n, $n > 0$, transversally at $q_k^n \sim x_k^n$, $0 \leq k \leq n$.

It is clear that X is a connected one-dimensional reducible analytic space. A holomorphic function f on X can be identified with a family $(f_n)_{n \geq 0}$ of holomorphic functions f_n on D_n, $n \geq 0$, such that $f_n(x_k^n) = f_0(q_k^n)$, $n > 0$, $0 \leq k \leq n$.

Let O_n be the image in X of the origin O of D_n, $n \geq 0$.

Theorem. (a) *The Carathéodory pseudodistance c_X is a distance on X, it defines the topology of X, and the space X is c_X-complete.*

(b) *There exists $r > 0$ such that the Carathéodory ball $B_{c_X}(O_0, r)$ is not relatively compact in X.*

Observe that by the Hurwitz theorem we have the following:

Lemma 1. *Let*

$$f_n(\lambda) := \prod_{k=0}^{n} \frac{\lambda - p_k^n}{1 - \overline{p_k^n}\lambda}, \qquad \lambda \in \Delta.$$

Suppose that $f_{n_j} \to f$ locally uniformly on Δ. Then $|f(0)| = 1/e$ and f is without zeros on Δ.

The natural isomorphism $i_n: D_n \to \Delta$ induces a holomorphic mapping p: $X \to D_0$.

Lemma 2. *Let $K \subset\subset \Delta$, and let N_0 be an integer such that $p_k^n \notin K$, $n \geq N_0$, $0 \leq k \leq n$. Denote by X_{N_0} the union of the images in X of the D_n, $n \geq N_0$. Then there exists a constant $C > 0$ such that, whenever x, $y \in X_{N_0}$, $x \neq y$, with $p(x) = p(y) \in K$, then $c_X(x, y) > C$.*

Proof. Suppose that $c_X(x_\nu, y_\nu) \to 0$, where x_ν, $y_\nu \in X$, $x_\nu \neq y_\nu$, and $p(x_\nu) = p(y_\nu) \in K$. Since bounded holomorphic functions on X separate points of X, we can assume that $x_\nu \in D_{n(\nu)}$ with $n(\nu) \to \infty$.

Observe that F_n with $F_n(x) := 0$, $x \in D_m$, $m \neq n$, and $F_n(x) := f_n(x)$, $x \in D_n$, is holomorphic on X. Therefore, since $c_X(x_\nu, y_\nu) \to 0$, we get that $f_{n(\nu)}(x_\nu) \to 0$, which contradicts Lemma 1. □

Now, we pass to the proof of the theorem.

Proof. Assertion (a) follows directly from Lemma 2. In order to prove (b), let $g: X \to \Delta$ be holomorphic with $g(O_0) = 0$. Define on Δ

$$\varphi_n(\lambda) := \tfrac{1}{2}(g(\lambda_n) - g(\lambda_0)),$$

where $\lambda_k \in X$ denotes the image in X of $\lambda \in D_k$, $k \geq 0$.

It is clear that $\varphi_n \in \mathscr{O}(\Delta, \Delta)$ and that $\varphi_n(p_k^n) = 0$, $0 \leq k \leq n$. Therefore, by the Schwarz Lemma, we obtain that

$$|\varphi_n(0)| \leq \left(1 - \frac{1}{1+n}\right)^{n+1},$$

which gives $|g(O_n)| \leq 2(1-\frac{1}{1+n})^{n+1}$. Hence $O_n \in B_{c_X}(O_0, r)$, $r := \omega(0, \frac{4}{5})$, if $n \gg 1$. $\square$

Observe that $B_{c_X}(O_0, r)$ is disconnected and has relatively compact components. Therefore the closure of $B_{c_X}(O_0, s)$ is not equal to the closed Carathéodory ball $\{y \in X : c_X(O_0, y) \leq s\}$ for some s (cf. [3]).

In order to conclude we recall that, so far, there is no example of an analytic space satisfying (a) of the theorem but which is not H^∞-convex.

References

1. M. Jarnicki and P. Pflug, *Invariant pseudodistances—completeness and product property*, Ann. Polon. Math. **55** (1991), 169–189.
2. M. Jarnicki, P. Pflug, and J.-P. Vigué, *The Carathéodory distance does not define the topology—the case of domains*, C. R. Acad. Sci. Paris Sér. I Math. **312** (1991), 77–79.
3. ——, *A remark on Carathéodory balls*, Arch. Math. **58** (1992), 595–598.
4. S. Kobayashi, *Intrinsic distances, measures and geometric function theory*, Bull. Amer. Math. Soc. **82** (1976), 357–416.
5. M. A. Selby, *On completeness with respect to the Carathéodory metric*, Canad. Math. Bull **17** (1974), 261–263.
6. N. Sibony, *Prolongement des fonctions holomorphes bornées et métrique de Carathéodory*, Invent. Math. **29** (1975), 205–230.
7. J.-P. Vigué, *The Carathéodory distance does not define the topology*, Proc. Amer. Math. Soc. **91** (1984), 223–224.

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