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SCHATTEN IDEAL BEHAVIOR OF A GENERALIZED HARDY OPERATOR

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ABSTRACT. We show the S^p-criteria and the cut-off result for an operator closely related to the classical Hardy inequality.

INTRODUCTION

For a function g defined on $(0, \infty)$ we consider the operator A_g acting on $L^2(0, \infty)$, which is given by the formula

$$A_g h(y) = \frac{g(y)}{y} \int_0^y h(x)\, dx.$$

We prove the S^p-criteria for A_g, $1 < p \leq \infty$, and the cut-off result at $p = 1$ which states that A_g cannot be trace class. This cut-off for A_g is related to the result of Rochberg in [R2]. In a more general sense it resembles the critical index behavior of commutators of singular integral operators found by Janson and Wolff in [JW] and the behavior of Hankel operators on Bergman spaces studied by Arazy, Fisher, and Peetre in [AFP]. The case $p = \infty$ is also contained in the paper [BK] by Bloom and Kerman. After this work was done the author was informed that closely related results had been obtained earlier by Birman and Solomyak in [BS].

The boundedness of A_g on $L^2(0, \infty)$ for $g \equiv 1$ becomes the classical Hardy inequality. The operator A_g is essentially a square root of a Calderón-Toeplitz operator T_μ defined in the context of the Haar wavelet, i.e.,

$$A_g^* A_g = P T_\mu P,$$

where $P : L^2(R) \to L^2(0, \infty)$ is the orthogonal projection, and also T_μ may be expressed in terms of $A_g^* A_g$. The symbol of T_μ is the measure μ defined on the upper half plane by the formula

$$\int_{-\infty}^{\infty} \int_0^{\infty} F(u, s)\, d\mu(u, s) = \int_0^{\infty} F(0, s) |g(s)|^2 \frac{ds}{s}.$$

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The background for Calderón-Toeplitz operators is presented in [R1, N]. The result in [N] for Calderón-Toeplitz operators shows that for a positive measure μ defined on the upper half plane and $p \geq 1$

$$T_\mu \in S^p \quad \text{if and only if} \quad \int_0^\infty \int_{-\infty}^\infty |\mu(D(u, s))|^p \, du \, \frac{ds}{s^2} < \infty,$$

where $D(u, s)$ is a hyperbolic disk with fixed radius centered at (u, s).

1. Preliminaries

For a compact operator T acting on a Hilbert space the nth singular value of T is defined as

$$s_n(T) = \inf\{||T - S|| : \operatorname{rank}(S) \leq n\}.$$

The Schatten ideal S^p, $1 \leq p < \infty$, is defined as the set of those compact operators T such that

$$||T||_{S^p} = \left(\sum_n (s_n(T))^p\right)^{1/p} < \infty.$$

In addition S^∞ stands for the algebra of bounded operators. We refer to [GK, Mc, Si, Z] for more information about Schatten ideals.

For g a measurable function defined on $(0, \infty)$

$$||g||_{l^p(L^2)} = \left(\sum_k \left(\int_{2^k}^{2^{k+1}} |g(x)|^2 \frac{dx}{x}\right)^{p/2}\right)^{1/p},$$

and $l^p(L^2)$ consists of those g for which $||g||_{l^p(L^2)}$ is finite.

A small letter c denotes a constant which changes its value from place to place. A symbol χ_A stands for the characteristic function of a set A.

Proposition 1.1 (see [Mc]). *Let $0 < p \leq 1$ and let T be a compact positive operator on a Hilbert space H. If for some orthonormal basis $\{\psi_i\}$ of H*

$$\sum_i |\langle T\psi_i, \psi_i\rangle|^p < \infty,$$

then $T \in S^p$ and

$$||T||_{S^p}^p \leq \sum_i |\langle T\psi_i, \psi_i\rangle|^p.$$

Proposition 1.2. *Let $1 \leq p \leq \infty$ and let $\{f_k\}$, $\{g_k\}$ be orthonormal sets in a Hilbert space H. If $T \in S^p$, then*

$$\left(\sum_k |\langle Tf_k, g_k\rangle|^p\right)^{1/p} \leq ||T||_{S^p}.$$

If T is compact, then

$$\lim_{|k|\to\infty} \langle Tf_k, g_k\rangle = 0.$$

Proof. The first part follows by a standard duality argument, and the second by an approximation argument.

Proposition 1.3. *Let* $1 \le p_0, p_1 \le \infty$, $0 \le \theta \le 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. *If* T *is a bounded operator from* $l^{p_0}(L^2)$ *into* S^{p_0} *and from* $l^{p_1}(L^2)$ *into* S^{p_1}, *then it is also bounded from* $l^p(L^2)$ *into* S^p *and*

$$||T||_{l^p(L^2)\to S^p} \le ||T||^{1-\theta}_{l^{p_0}(L^2)\to S^{p_0}} ||T||^{\theta}_{l^{p_1}(L^2)\to S^{p_1}}.$$

Proof. The proof follows from standard interpolation results; in particular, Theorem 5.1.2 in [BL] and Theorem 4.2.7 in [Z].

2. The main result

Theorem 2.1. *Let* $1 < p \le \infty$ *and let* g *be a locally square integrable function.*

(1) *The operator* A_g *is in the Schatten ideal* S^p *if and only if* g *belongs to* $l^p(L^2)$; *moreover, the corresponding norms are equivalent. The operator* A_g *is compact if and only if*

$$\lim_{|k|\to\infty} \int_{2^k}^{2^{k+1}} |g(y)|^2 \frac{dy}{y} = 0.$$

(2) *If* $A_g \in S^1$, *then* $g \equiv 0$.

Condition (2) states that the family of operators $\{A_g\}$ is "cut off" at $p = 1$.

Proof. We prove that the map A, $A(g) = A_g$, is bounded from $l^\infty(L^2)$ into S^∞ and from $l^p(L^2)$ into S^p, $1 < p \le 2$. It follows by Proposition 1.3 that it is bounded from $l^p(L^2)$ into S^p for all $1 < p \le \infty$.

We first consider $p = \infty$. We observe that

$$\int_0^\infty \left| \frac{g(y)}{y} \int_0^y h(x)\,dx \right|^2 dy = \sum_k \int_{2^k}^{2^{k+1}} \frac{|g(y)|^2}{y^2} \left(\int_0^y |h(x)|\,dx \right)^2$$
$$\le \sup_k \int_{2^k}^{2^{k+1}} \frac{|g(y)|^2}{y}\,dy \sum_k \frac{1}{2^k} \left(\int_0^{2^{k+1}} |h(x)|\,dx \right)^2$$

and that

$$\left(\sum_k \frac{1}{2^k} \left(\int_0^{2^{k+1}} |h(x)|\,dx \right)^2 \right)^{1/2}$$
$$\le c \left(\int_0^\infty \left(\frac{1}{y} \int_0^y |h(x)|\,dx \right)^2 dy \right)^{1/2} \le c||h||,$$

where the last inequality is justified by the classical Hardy inequality. The above computation shows that A is bounded from $l^\infty(L^2)$ into S^∞.

Now let $1 < p \le 2$. It is clear that $A_g^* A_g$ is given by the kernel

$$K(x, z) = \int_0^\infty \frac{|g(y)|^2}{y^2} \chi_{[0,y]}(x) \chi_{[0,y]}(z)\,dy.$$

Let

$$K_k(z, x) = \int_{2^k}^{2^{k+1}} \frac{|g(y)|^2}{y^2} \chi_{[0,y]}(x) \chi_{[0,y]}(z)\,dy.$$

The operator given by K_k acts on $L^2(0, 2^{k+1})$, and

$$e_n^k(x) = 2^{-(k+1)/2} e^{2\pi i 2^{-(k+1)} nx}$$

forms an orthonormal basis on that space. A direct computation gives

$$|\langle K_k e_n^k, e_n^k \rangle| \leq \frac{c}{n^2} \int_{2^k}^{2^{k+1}} |g(y)|^2 \frac{dy}{y}.$$

We obtain by Proposition 1.1

$$||K_k||_{S^{p/2}}^{p/2} \leq \sum_n |\langle K_k e_n^k, e_n^k \rangle|^{p/2} \leq c \left(\int_{2^k}^{2^{k+1}} |g(y)|^2 \frac{dy}{y} \right)^{p/2},$$

and this provides the estimate

$$||A_g||_{S^p}^p = ||K||_{S^{p/2}}^{p/2} \leq \sum_k ||K_k||_{S^{p/2}}^{p/2} \leq c||g||_{l^p(L^2)}^p.$$

Thus A is bounded from $l^p(L^2)$ into S^p, $1 < p \leq 2$.

It is easy to observe at this point that if

$$\lim_{|k| \to \infty} \int_{2^k}^{2^{k+1}} |g(y)|^2 \frac{dy}{y} = 0,$$

then A_g is compact.

Now for the converse estimate we assume that $A_g \in S^p$, and we take

$$f_k(x) = 2^{-(k-1)/2} \chi_{[2^{k-1}, 2^k]}(x),$$

$$g_k(y) = \frac{g(y)}{y} \chi_{[2^k, 2^{k+1}]}(y) \left(\int_{2^k}^{2^{k+1}} \frac{|g(y)|^2}{y^2} dy \right)^{-1/2}.$$

It follows by Proposition 1.2 that

$$\sum_k |\langle A_g f_k, g_k \rangle|^p \leq ||A_g||_{S^p}^p.$$

A direct computation gives

$$\langle A_g f_k, g_k \rangle \cong \left(\int_{2^k}^{2^{k+1}} |g(y)|^2 \frac{dy}{y} \right)^{1/2},$$

and this proves the converse estimate.

Now we discuss the cut-off. Assume that for some $0 < a < b < \infty$

$$0 < \int_a^b |g(y)|^2 \frac{dy}{y} < \infty$$

and that $A_g \in S^1$. The kernel

$$B_g(y, x) = \chi_{(a,b)}(y) \frac{g(y)}{y} \chi_{[0,y]}(x) \chi_{(0,b)}(x)$$

defines an operator in $S^1(0, b)$. Let $e_n(x) = e^{2\pi i b n x}$. It is easy to check that

$$\langle B_g e_n, e_n \rangle = c(\hat{g}_0(0) - \hat{g}_0(n))/n,$$

where $g_0(y) = \frac{g(y)}{y}\chi_{(a,b)}(y)$. Changing a, b if necessary we may arrange that

$$\hat{g}_0(0) = \int_a^b \frac{g(y)}{y}\,dy \neq 0.$$

By the Riemann-Lebesgue lemma $\hat{g}_0(n) \to 0$, so

$$\sum_n |\langle B_g e_n, e_n\rangle| = \infty,$$

and this contradicts Proposition 1.2. Thus A_g is not in the trace class.

Remark 2.2. The S^p-criteria for A_g, $p \geq 2$, may also be obtained by an application of the results for Calderón-Toeplitz operators in [N].

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References

[AFP] J. Arazy, S. Fisher, and J. Peetre, *Hankel operators on weighted Bergman spaces*, Amer. J. Math. **110** (1988), 989–1055.

[BK] S. Bloom and R. Kerman, *Weighted norm inequalities for operators of Hardy type*, Proc. Amer. Math. Soc. **113** (1991), 135–141.

[BL] J. Bergh and J. Löfström, *Interpolation spaces: An introduction*, Grundlehren Math. Wiss., vol. 223, Springer-Verlag, Berlin, 1976.

[BS] M. Sh. Birman and M. Z. Solomyak, *Estimates for the number of negative eigenvalues of the Schrödinger operator and its generalizations*, Adv. Soviet Math. **7** (1991), 1–55.

[FS] J. J. F. Fournier and J. Stewart, *Amalgams of L^p and l^q*, Bull. Amer. Math. Soc. (N.S.) **13** (1985), 1–21.

[GK] I. Gohberg and M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Transl. Math. Monographs, vol. 18, Amer. Math. Soc., Providence, RI, 1969.

[JW] S. Janson and T. Wolff, *Schatten classes and commutators of singular integral operators*, Ark. Mat. **82** (1982), 301–310.

[Mc] C. A. McCarthy, c_p, Israel J. Math. **5** (1967), 249–271.

[N] K. Nowak, Ph.D. thesis, Washington Univ. in St. Louis, MO, 1991.

[R1] R. Rochberg, *Toeplitz and Hankel operators, wavelets, NWO sequences, and almost diagonalization of operators* (W. B. Arveson and R. G. Douglas, eds.), Proc. Sympos. Pure Math., vol. 51, Amer. Math. Soc., Providence, RI, 1990, pp. 425–444.

[R2] ——, *Eigenvalue estimates for Calderón-Toeplitz operators*, preprint, 1990.

[Si] B. Simon, *Trace ideals and their applications*, Cambridge Univ. Press, London and New York, 1979.

[Z] K. Zhu, *Operator theory in function spaces*, Marcel Dekker, New York, 1990.

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