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ON GENERALIZED MAXIMAL FUNCTIONS

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ABSTRACT. In this paper we study the question of under what circumstances the quantity $\| \sup_{t<\infty, a\in\mathbf{R}} | \int_0^t f(a, M_s)\, dM_s| \|_p$ is comparable to $\|M_\infty^*\|_p$, where M_t is a continuous martingale and f is a bounded Borel-measurable function.

1. INTRODUCTION

Let $\{M_t\}_{t\geq 0}$ be a continuous martingale that starts at 0 and satisfies the "usual conditions". Let $M_t^* := \sup_{s\leq t} |M_s|$ be the usual maximal function and let $L_t^* := \sup_{a\in\mathbf{R}} L_t^a$ be the maximal function of the local time $L_t^a := (M_t - a)^+ - (M_0 - a)^+ - \int_0^t 1_{M_s>a}\, dM_s$ of M_t. Barlow and Yor showed in [2, 3] that for all $p > 0$ we have that $\|M_\infty^*\|_p$ is comparable to $\|L_\infty^*\|_p$, i.e., that there are universal constants c, $C > 0$ only depending on p such that $c\|M_\infty^*\|_p \leq \|L_\infty^*\|_p \leq C\|M_\infty^*\|_p$. It can be shown that for $p > 0$ we have that $\| \sup_{t<\infty, a\in\mathbf{R}} | \int_0^t 1_{M_s>a}\, dM_s| \|_p$ is comparable to $\|M_\infty^*\|_p$. This immediately leads to the following question: Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a bounded Borel-measurable function. Under what circumstances is $\| \sup_{t<\infty, a\in\mathbf{R}} | \int_0^t f(a, M_s)\, M_s| \|_p$ comparable to $\|M_\infty^*\|_p$? We will give a sufficient condition (cf. Theorem 2.3) for this to happen, which enables us to give a wide variety of examples.

2. RESULTS

In this section we define a generalized maximal function for a continuous martingale (cf. Definition 2.1) and investigate when norm inequalities of the proposed type hold. In Theorem 2.3(i) we give a condition that ensures that the generalized maximal function is progressively measurable. Part (ii) is a lemma that also enables us to give the variety of examples as shown in Example 2.4. In part (iii) we give a sufficient condition for "upper comparability" that—in light of Example 2.5—appears to be fairly sharp. Finally, in part (iv) we give a sufficient condition for "lower comparability".

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Definition 2.1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a bounded Borel-measurable function. Then we can define

$$Mf_t^a := \int_0^t f(a, M_s)\, dM_s$$

and we define the *generalized maximal function of* M_t *associated with* f to be

$$Mf_t^* := \operatorname*{ess\,sup}_{s\le t,\, a\in\mathbf{R}} |Mf_s^a|.$$

When there could be doubt as to which martingale a generalized maximal function belongs to we will write $Mf(M_u)_t^a$ for Mf_t^a, etc.

Definition 2.2. Let $\hat{p} > 0$. Then f satisfies the *$\hat{p}$-continuity condition* iff there is a $c > 0$ such that for all a, $b \in \mathbb{R}$ we have that

$$\int_{\mathbf{R}} |f(a, x) - f(b, x)|^{\hat{p}}\, dx \le c|b-a|.$$

f satisfies the *cone condition* iff there is a $C > 0$ such that for all $a \in \mathbb{R}$ we have that

$$V_{-a}^a f(a, \cdot) \le C,$$

where $V_{-a}^a f(a, \cdot)$ denotes the variation of the function $x \mapsto f(a, x)$ over the interval $[-a, a]$.

Theorem 2.3. *Let $\hat{p} \ge 2$ and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a bounded Borel-measurable function that satisfies the $\hat{p}$-continuity condition. Then*

(i) *Mf_t^a has a continuous version, i.e., there is a process $\widehat{M}f_t^a$ that is continuous in (a, t) such that $\widehat{M}f_t^a = Mf_t^a$ a.s. Therefore Mf_t^* is progressively measurable.*

(ii) *$g(a, x) := f(a, x)1_{|a|<|x|}$ satisfies the $\hat{p}$-continuity condition and the cone condition.*

(iii) *If f satisfies the cone condition, then for $0 < p < \infty$ there is a $C_p > 0$ such that for all continuous martingales M_t we have that*

$$\|Mf_\infty^*\|_p \le C_p \|M_\infty^*\|_p.$$

(iv) *If there is an $\varepsilon > 0$ such that for each $N > 0$ there is an $a \in \mathbb{R}$ such that $f(a, \cdot)|_{[-N, N]} > \varepsilon$, then for $0 < p < \infty$ there is a $c_p > 0$ such that for all continuous martingales M_t we have that*

$$\|Mf_\infty^*\|_p \ge c_p \|M_\infty^*\|_p.$$

Example 2.4. Let $\hat{p} \ge 2$ and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a bounded Borel-measurable function that satisfies the $\hat{p}$-continuity condition. Let $\tilde{f} := f(a, x)1_{|a|<|x|} + 1_{a\le -|x|}$. Then for $0 < p < \infty$ we have that there are constants c_p, $C_p > 0$ such that for all continuous martingales M_t:

$$c_p\|M_\infty^*\|_p \le \|M\tilde{f}_\infty^*\|_p \le C_p\|M_\infty^*\|_p.$$

Example 2.5. Let

$$\mathscr{F} := \Big\{ g\colon \mathbb{R} \to \{-1, 0, 1\} \,|\, \exists n \in \mathbb{N} : \forall k \in \{-n+1, \dots, n\}\colon g \text{ is constant and } |g| = 1 \text{ on } \left[\frac{k-1}{n}, \frac{k}{n}\right),\ \text{and } g|_{\mathbf{R}\setminus[-1,1)} = 0 \Big\}.$$

Let $\{g_n\}_{n\in\mathbb{N}}$ be a counting of $\mathscr{F}$. Define

$$f(a,x) := \begin{cases} g_{a/2}(x) & \text{if } a \in \mathbb{N} \text{ is even},\\ 0 & \text{if } a \in \mathbb{N} \text{ is odd or } a \le 0,\\ (1-(a-[a]))f([a],x) & \\ \quad +(a-[a])f([a]+1,x) & \text{otherwise}. \end{cases}$$

Then f satisfies the $\hat{p}$-continuity condition for $\hat{p} = 2$ and it obviously does not satisfy the cone condition. In can be shown (cf. [7]) that $Mf_t^* = \infty$ on a set of positive measure for $t > 0$ and for M_t Brownian motion stopped at a time $T > 0$.

3. The proof

Throughout this proof let M_t be a continuous martingale and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a bounded Borel-measurable function that satisfies the $\hat{p}$-continuity condition for some $\hat{p} \ge 2$. Let $\hat{q}$ be such that $1/\hat{p} + 1/\hat{q} = 1$, and let $T > 0$. Recall that for $x > 0$ we have that $\tau_x := \inf\{t > 0\colon |M_t| = x\}$ is a stopping time.

(a) *Mf_t^a has a version $\widehat{M}f_t^a$ that is jointly continuous in (a,t). Thus Mf_t^a is progressively measurable and since $Mf_t^* = \widehat{M}f_t^*$ a.s. we are free to assume without loss of generality that Mf_t^a is continuous. Moreover there is a constant $C > 0$ depending on $\hat{p}$ and $p' \ge \hat{q}$ only such that for all $\lambda > 0$, and $r, s \in \mathbb{R}$ we have that*

$$E\left(\sup_{t\le T} |Mf_t^r - Mf_t^s|^{p'}\,;\, M_T^* \le \lambda\right) \le C|r-s|^{p'/\hat{p}} E((M_T^* \wedge \lambda)^{p'/\hat{p}}).$$

An application of [2, Theorem 4.1] shows that

$$\begin{aligned} &\left\|\sup_{t\le T} |Mf_t^a - Mf_t^b|\right\|_{p'} \\ (*)\qquad &\le C' \left\| \left\langle \int_0^{\cdot} (f(a,M_t) - f(b,M_t))\,dM_t \right\rangle_T^{1/2} \right\|_{p'} \\ &\le C''\|f(a,\cdot) - f(b,\cdot)\|_{\hat{p}} \|\langle M\rangle_T^{1/2}\|_{p'/\hat{q}}^{1/q} \qquad \text{(by [2, Theorem 4.1])} \\ &\le C'''|b-a|^{1/\hat{p}} \|\langle M\rangle_T^{1/2}\|_{p'/\hat{q}}^{1/\hat{q}}. \end{aligned}$$

To prove the assertions regarding continuity, let $p' > \hat{p}$, $\hat{q}$ and let $N > 0$. Clearly $\|\langle M\rangle_{t\wedge\tau_N}\|_{p'/\hat{q}} < \infty$. Thus there is a constant $\overline{C} > 0$ such that

$$E\left(\sup_{s\le t} |Mf_{s\wedge\tau_n}^a - Mf_{s\wedge\tau_n}^b|^{p'}\right) \le \overline{C}|b-a|^{p'/\hat{p}}.$$

An application of Kolmogorov's Theorem on continuous processes now yields that $Mf_{t\wedge\tau_N}^a$ has a jointly continuous version. Since $\tau_N \to \infty$ for $N \to \infty$ this implies that Mf_t^a has a jointly continuous version. An easy computation now shows that Mf_t^* is progressively measurable and $Mf_t^* = \widehat{M}f_t^*$ a.s.

Finally for $t \le T$ on $\{M_T^* \le \lambda\} = \{T \le \tau_\lambda\}$ we a.s. have that

$$Mf(M_u)_t^r = \int_0^t f(r, M_s)\, dM_s = \int_0^{t\wedge\tau_\lambda} f(r, M_{s\wedge\tau_\lambda})\, dM_{s\wedge\tau_\lambda}$$
$$= \int_0^t f(r, M_{s\wedge\tau_\lambda})\, dM_{s\wedge\tau_\lambda} = Mf(M_{u\wedge\tau_\lambda})_t^r,$$

and therefore

$$E\left(\sup_{t\le T} |Mf(M_u)_t^r - Mf(M_u)_t^s|^{p'} ; M_T^* \le \lambda\right)$$
$$\le E\left(\sup_{t\le T} |Mf(M_{u\wedge\tau_\lambda})_t^r - Mf(M_{u\wedge\tau_\lambda})_t^s|^{p'}\right)$$
$$\le C|r-s|^{p'/\hat{p}} E((M_{T\wedge\tau_\lambda}^*)^{p'/\hat{q}}) \quad \text{(by } (*))$$
$$= C|r-s|^{p'/\hat{p}} E((M_T^* \wedge \lambda)^{p'/\hat{q}}).$$

(b) *Suppose that f is such that for $|a| \ge |x|$ we have that $f(a, x) = 0$. Then for all $\lambda > 0$ there is a nullset $S \subseteq \{M_T^* \le \lambda\}$ such that for all $\omega \in \{M_T^* \le \lambda\} \setminus S$ we have that $Mf_t^r = 0$ for $|r| \ge \lambda$.*

First notice that by continuity of Mf_t^r it is enough to prove that $Mf_t^r = 0$ for $|r| \ge \lambda$ with t, $r \in \mathbb{Q}$. For $t \le T$ and $|r| \ge \lambda$ there is a nullset $S_{r,t}$ such that for $\omega \in \{M_T^* \le \lambda\} \setminus S_{r,t}$:

$$Mf_t^r = Mf_{t\wedge\tau_\lambda}^r = \int_0^{t\wedge\tau_\lambda} f(r, M_u)\, dM_u = \int_0^{t\wedge\tau_\lambda} 0\, dM_u = 0.$$

Thus by continuity of Mf_t^r we have on $\{M_T^* \le \lambda\} \setminus \bigcup_{|r|\ge\lambda, r\in\mathbb{Q}, t\le T, t\in\mathbb{Q}} S_{r,t}$ that $Mf_t^r = 0$ for $|r| \ge \lambda$.

(c) *Suppose that f is such that for $|a| \ge |x|$ we have that $f(a, x) = 0$. Then there is a $C > 0$ depending only on p such that*

$$\|Mf_\infty^*\|_p \le C\|M_\infty^*\|_p.$$

This part of the proof was inspired by the work of Gundy and Silverstein in [5]. It is completely analogous to the conclusion of the proof in [5, §4]. Recall from [5] the following version of the Garsia-Rodemich-Rumsey-Lemma (GRR): If a continuous function $F: I = [a, b] \to \mathbb{R}$ has a zero in I, then for $q > 0$, $p' > \frac{2}{q}$ we have that

$$\sup_{r\in I} |F(r)| \le C|I|^q \left\{ \frac{1}{|I|^2} \iint_{I\times I} \left| \frac{F(r) - F(s)}{|r-s|^q} \right|^{p'} dr\, ds \right\}^{1/p'}$$

where C is a universal constant.

Also recall from [5] that if there are $p' > p$ and C, $D > 0$ such that for all $\lambda > 0$ we have that

$$P(G \ge \lambda) \le \frac{C}{\lambda^{p'}} \int_{\{H\le\lambda\}} H^{p'}\, dP + DP(H > \lambda),$$

then

$$\int_\Omega G^p\, dP \le \left(C\frac{p}{p'-p} + D\right) \int_\Omega H^p\, dP.$$

Let $p' > \max\{2\hat{p}, p\hat{q}\}$ and let S be as in (b). By GRR with $I := [-\lambda, \lambda]$ for $\omega \in \{M_T^* \leq \lambda\} \setminus S$ we have that

$$\begin{aligned}(Mf_T^*)^{p'} &= \sup_{t\leq T}\sup_{|r|\leq\lambda} |Mf_t^r|^{p'} \quad \text{(by (b))} \\ &\leq \sup_{t\leq T} C\lambda^{p'/\hat{p}}\frac{1}{(2\lambda)^2}\int_{I\times I}\left(\frac{|Mf_t^r - Mf_t^s|}{|r-s|^{1/\hat{p}}}\right)^{p'} dr\,ds \quad \text{(by GRR)} \\ &\leq C\lambda^{p'/\hat{p}}\frac{1}{(2\lambda)^2}\int_{I\times I}\left(\frac{\sup_{t\leq T}|Mf_t^r - Mf_t^s|}{|r-s|^{1/\hat{p}}}\right)^{p'} dr\,ds.\end{aligned}$$

Integration now yields via (a)

$$E((Mf_T^*)^{p'}\,;\, M_T^* \leq \lambda) \leq C'\lambda^{p'/\hat{p}}E((M_T^* \wedge \lambda)^{p'/\hat{q}}).$$

Therefore by Chebyshev's inequality:

$$\begin{aligned}P(Mf_T^* > \lambda) &\leq P(Mf_T^* > \lambda,\ M_T^* \leq \lambda) + P(M_T^* > \lambda) \\ &\leq \frac{1}{\lambda^{p'}}E((Mf_T^*)^{p'}\,;\, M_T^* \leq \lambda) + P(M_T^* > \lambda) \\ &\leq \frac{C'}{\lambda^{p'(1-1/\hat{p})}}E((M_T^* \wedge \lambda)^{p'/\hat{q}}) + P(M_T^* > \lambda) \\ &\leq \frac{C'}{\lambda^{p'/\hat{q}}}E((M_T^*)^{p'/\hat{q}}\,;\, M_T^* \leq \lambda) + (C'+1)P(M_T^* > \lambda).\end{aligned}$$

Using the above mentioned good-λ-inequality we obtain

$$E((Mf_T^*)^p) \leq C''E((M_T^*)^p).$$

(d) *For the usual local time* $L_t^a = (M_t - a)^+ - (M_0 - a)^+ - \int_0^t 1_{M_s > a}\,dM_s$ *of* M_t *and* $p > 0$ *there is a* $C_p > 0$ *that only depends on* p *such that*

$$\|L_T^*\|_p \leq C_p\|M_T^*\|_p.$$

Clearly

$$L_T^* \leq ((M_t - a)^+ - (M_0 - a)^+)_T^* + \left(\int_0^t 1_{M_s > a}\,dM_s\right)_T^*,$$

where $X(t, a)_T^* = \sup_{t\leq T, a\mathbf{R}}|X(t, a)|$. Moreover

$$((M_t - a)^+ - (M_0 - a)^+)_T^* \leq M_T^*.$$

To see that there is a constant $C > 0$ only depending on p such that

$$\left\|\left(\int_0^t 1_{M_s > a}\,dM_s\right)_T^*\right\|_p \leq C\|M_T^*\|_p,$$

notice that part (c) works for $f(a, x) := 1_{x>a}$ since this function satisfies the $\hat{p}$-continuity condition for any $\hat{p} > 0$ and since with arguments as in (b) one can show that on $\{M_T^* \leq \lambda\}$ we a.s. have that $M(1_{x>a})_t^r = 0$ for $r \geq \lambda$ and $M(1_{x>a})_t^r = M_t$ for $r \leq -\lambda$. Thus the above also provides a new short proof of the result of Barlow and Yor.

(e) *Let* $g(a, x) := f(a, x)1_{|a|<|x|}$ *and let* $h(a, x) := f(a, x)1_{|a|\geq|x|}$. *Then* g *and* h *satisfy the* $\hat{p}$*-continuity condition.*

We only give the proof for h, since the proof for g is similar. Let f be bounded by, say, C, let a, $b \in \mathbb{R}$ be fixed, and let $a \leq b$. Then

$$\begin{aligned}
&\|h(a, \cdot) - h(b, \cdot)\|_{\hat{p}} \\
&\quad \leq \|f(a, \cdot)1_{|a|\geq|\cdot|} - f(b, \cdot)1_{|a|\geq|\cdot|}\|_{\hat{p}} + \|f(b, \cdot)1_{|a|\geq|\cdot|} - f(b, \cdot)1_{|b|\geq|\cdot|}\|_{\hat{p}} \\
&\quad = \|(f(a, \cdot) - f(b, \cdot))1_{|a|\geq|\cdot|}\|_{\hat{p}} + \|f(b, \cdot)(1_{|a|\geq|\cdot|} - 1_{|b|\geq|\cdot|})\|_{\hat{p}} \\
&\quad \leq \|f(a, \cdot) - f(b, \cdot)\|_{\hat{p}} + C\|(1_{|a|\geq|\cdot|} - 1_{|b|\geq|\cdot|})\|_{\hat{p}} \leq C'|b-a|^{1/\hat{p}}.
\end{aligned}$$

(f) *Suppose that* f *satisfies the cone condition and let* h *be as in* (e). *Then there is a constant* $C_p > 0$ *that only depends on* p *such that*

$$\|Mh_T^*\|_p \leq C_p\|M_T^*\|_p.$$

An application of the Hô-Tanaka formula shows that for all $a \in \mathbb{R}$:

$$\frac{1}{2}\int_{\mathbf{R}} L_t^x\, dh(a, x) = \int_{M_0}^{M_t} h(a, y)\, dy - \int_0^t h(a, M_s)\, dM_s \quad \text{a.s.}$$

Therefore by the continuity of Mh_t^r we have that

$$\begin{aligned}
Mh_T^* &= \sup_{t\leq T, a\in\mathbf{R}} \left|\int_0^t h(a, M_s)\, dM_s\right| \\
&\leq \sup_{t\leq T, a\in\mathbf{R}} \left|\int_{M_0}^{M_t} h(a, y)\, dy\right| + \sup_{t\leq T, a\in\mathbf{R}} \left|\frac{1}{2}\int_{\mathbf{R}} L_t^x\, dh(a, x)\right| \\
&\leq \sup_{t\leq T, a\in\mathbf{R}} \left|\int_{M_0}^{M_t} |h(a, y)|\, dy\right| + \frac{1}{2}\sup_{a\in\mathbf{R}} V_{-a}^a h(a, \cdot) \sup_{t\leq T, a\in\mathbf{R}} |L_t^x| \\
&\leq CM_T^* + CL_T^*.
\end{aligned}$$

Proof of Theorem 2.3. Parts (i) and (ii) are proved in (a) and (e), respectively. To prove part (iii) let g and h be as in (e). Since g satisfies the hypothesis of (c) we have that there is a $C > 0$ that only depends on p such that $\|Mg_T^*\|_p \leq C\|M_T^*\|_p$. Combining this with (f) we obtain:

$$\|Mf_T^*\|_p \leq \|Mg_T^*\| + \|Mh_T^*\|_p \leq C_p\|M_T^*\|_p.$$

Now let T go to infinity.

To prove (iv) for $N > 0$ let a_N be such that $f(a_N, \cdot)|_{[-N, N]} < \varepsilon$. Then

$$\begin{aligned}
E&\left(\sup_{a\in\mathbf{R}, t>0}\left|\int_0^t f(a, M_s)\, dM_s\right|^p\right) \geq E\left(\sup_{a\in\mathbf{R}, t>0}\left|\int_0^{t\wedge\tau_N} f(a, M_s)\, dM_s\right|^p\right) \\
&\geq E\left(\left(\left(\int_0^{\cdot\wedge\tau_N} f(a_N, M_s)\, dM_s\right)^*_\infty\right)^p\right) \\
&\geq C'E\left(\left(\int_0^{\tau_N} f^2(a_N, M_s)\, d\langle M\rangle_s\right)^{p/2}\right) \geq C'\varepsilon^2 E(\langle M\rangle_{\tau_N}^{p/2}).
\end{aligned}$$

Now let N go to infinity.

Proof of Example 2.4. It is easy to see that $\tilde{f}$ satisfies the conditions in (iii) and (iv) of Theorem 2.3.

Proof of Example 2.5. The proof is tedious but not very hard and can be found in [7].

4. Conclusion

We have defined generalized maximal functions and shown that a wide variety of generalized maximal functions satisfy norm inequalities of the Barlow-Yor type. The results presented here complement the results that Bass and Davis obtained in [1, 4]. In [1, 4] it is shown that Barlow-Yor-like inequalities hold for functionals that have scaling properties similar to the scaling properties of local time, while our results show that Barlow-Yor-like results hold for generalized maximal functions Mf_t^a for which f has geometric properties that are similar to the properties of $1_{x>a}$. It is worth pointing out that the results presented here are not accessible from the results in [1, 4].

An interesting problem that will be the object of our further research is how to define a generalized density functional of the area integral (cf. [5]).

Results on generalized maximal functions for semimartinagles, and martingales that do not start at 0, and the relation of these results to the results presented in [1] can be found in [7]. In [7] we also consider the slightly more general case where f need not be bounded.

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