

## ON THE NUMBER OF DTr-ORBITS CONTAINING DIRECTING MODULES

PENG LIAN-GANG AND XIAO JIE

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**ABSTRACT.** In this note we prove that all directing modules over an artin algebra are distributed to finitely many DTr-orbits.

### 1. INTRODUCTION

Let us first recall some basic notation in the representation theory of algebras. Throughout this note  $A$  denotes an artin algebra over a commutative artin ring  $R$ , that is,  $A$  is an  $R$ -algebra that is finitely generated as an  $R$ -module. All  $A$ -modules that we consider are finitely generated left modules. We denote by  $\text{mod } A$  the category of finitely generated left  $A$ -modules and by  $\text{ind } A$  the full subcategory of  $\text{mod } A$  consisting of all indecomposable modules.  $\Gamma_A$  is the Auslander-Reiten quiver of  $A$ . We use  $\tau$  to denote the Auslander-Reiten operator DTr.

For  $M, N \in \text{ind } A$ ,  $N$  is said to be a successor (respectively, a proper successor) of  $M$ , denoted by  $M \leq N$  (respectively,  $M < N$ ), if there exists a chain

$$M = M_0 \xrightarrow{f_1} M_1 \rightarrow \cdots \rightarrow M_{s-1} \xrightarrow{f_s} M_s = N$$

in  $\text{ind } A$ , where every  $f_i$  is nonzero (respectively,  $s > 0$  and every  $f_i$  is nonzero and nonisomorphic). We call an indecomposable  $A$ -module  $M$  directing if  $M \not\prec M$ , that is,  $M$  does not belong to any cycle of nonzero and nonisomorphic morphisms in  $\text{ind } A$ .

Skowroński-Smalø [SS] proved that if  $\mathcal{C}$  is a component of  $\Gamma_A$  consisting entirely of directing modules then  $\mathcal{C}$  has only finitely many  $\tau$ -orbits; the number of these kind of components is also finite. This note is devoted to getting a more general result. We prove that all directing  $A$ -modules only belong to finitely many  $\tau$ -orbits. In other way it is not difficult to give an example of artin algebras over which some directing modules, as well as nondirecting modules, are contained in one component of its Auslander-Reiten quiver.

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2. SOME PROPOSITIONS AND THE THEOREM

**Proposition 2.1.** *Let  $M$  be an indecomposable  $A$ -module. If  $M$  is not directing and no indecomposable projective  $A$ -module is a successor of  $M$ , then  $M \leq \tau M$ .*

*Proof.*  $M$  is not projective by the assumption. Since  $M$  is not directing, there is a chain of nonzero and nonisomorphic morphisms

$$M = M_0 \xrightarrow{f_1} M_1 \rightarrow \cdots \rightarrow M_{s-1} \xrightarrow{f_s} M_s = M$$

in  $\text{ind } A$ , where  $s > 0$ . Obviously any indecomposable projective  $A$ -module is not a successor of  $M_i$  for all  $i$ . Therefore  $M_i$  is not projective and  $\text{inj dim } \tau M_i \leq 1$  for all  $i$  according to [R, 2.4 (1\*)]. If every  $f_i$  is irreducible, then there exists some  $M_j$  such that  $M \leq M_j \leq \tau M_j \leq \tau M$  by [BS]. Thus we can assume some  $f_t$  is not a linear combination of compositions of irreducible maps but  $f_{t+1}, \dots, f_s$  is irreducible. We may further assume  $s - t \geq n$ , where  $n$  is the number of all nonisomorphic simple  $A$ -modules. If there exist  $i$  and  $j$  satisfying  $t \leq i, j \leq s, i \neq j$  but  $M_i \simeq M_j$ , then we know  $M \leq M_i \leq \tau M_i \leq \tau M$  by [BS]. So we assume that  $M_t, M_{t+1}, \dots, M_s$  are pairwise nonisomorphic. The dual of the well-known Bongartz Lemma (see [B]) tells us

$$0 \neq \text{Ext}_A^1 \left( \tau \left( \bigoplus_{i=t}^s M_i \right), \tau \left( \bigoplus_{i=t}^s M_i \right) \right) \simeq D \underline{\text{Hom}}_A \left( \bigoplus_{i=t}^s M_i, \bigoplus_{i=t}^s \tau M_i \right).$$

Therefore there exist  $M_i$  and  $M_j$  such that  $M_i \leq \tau M_j$ . This also implies  $M \leq \tau M$ . Q.E.D.

**Definition 2.2.** Let  $M$  be an indecomposable  $A$ -module.  $M$  is called strongly directing if  $M$  is directing and any indecomposable projective  $A$ -module is not a proper successor of  $M$ .

**Corollary 2.3.** *Let  $M$  be a strongly directing  $A$ -module. If there is a chain of irreducible maps from  $M$  to  $N$  then  $N$  is also strongly directing.*

*Proof.* Let  $M = M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_s$  be a chain of irreducible maps where each  $M_i$  is indecomposable. It is sufficient to prove that  $M_s$  is directing. If  $M_s$  is not directing, since any indecomposable projective module is not a successor of  $M_s$ , then  $M_s \leq \tau M_s$  by Proposition 2.1. Thus  $M_{s-1} < M_s \leq \tau M_s < M_{s-1}$ , that is,  $M_{s-1}$  is also not directing. Inductively, we will arrive at a contradiction that asserts that  $M = M_0$  is not directing. Consequently, we know that  $M_s$  must be directing. Q.E.D.

**Corollary 2.4.** *Let  $\mathcal{E}$  be a component of  $\Gamma_A$  and  $M \in \mathcal{E}$  be a strongly directing  $A$ -module. Then the number of  $\tau$ -orbits in  $\mathcal{E}$  containing successors of  $M$  with respect to irreducible maps is finite.*

*Proof.* Assume that the number of the  $\tau$ -orbits that contain some successors of  $M$  with respect to irreducible maps is infinite. Then it is easy to see that there are successors  $X$  and  $Y$  of  $M$  with respect to irreducible maps satisfying the following conditions.

- (a) In the orbit graph of  $\Gamma_A$ , the length of the shortest path between  $[X]$  and  $[Y]$  is greater than  $n$ , where  $n$  is the number of all nonisomorphic simple  $A$ -modules.

(b) There exists a chain of irreducible maps

$$X = N_0 \rightarrow N_1 \rightarrow \dots \rightarrow N_m = Y$$

where each  $N_i$  is indecomposable, there exists no injective module belonging to the  $\tau$ -orbits of  $N_i$ , and  $N_0, N_1, \dots, N_m$  belong to the distinct  $\tau$ -orbits.

Now we put  $H_i = \tau^{-(m-i)}N_i$ . Then there is a chain of irreducible maps

$$Y = H_m \rightarrow H_{m-1} \rightarrow \dots \rightarrow H_1 \rightarrow H_0 = \tau^{-m}X.$$

Since  $m > n$ , by a similar method as in the proof of Proposition 2.1, we can see  $Y \leq \tau H_0$ . We next show that  $Y \leq \tau^t H_0$  for any  $0 \leq t \leq m$ . Suppose that there exists  $0 < t < m$  such that  $Y \leq \tau^t H_0$ . Thus there exists a chain of nonzero and nonisomorphic morphisms

$$Y = Z_0 \xrightarrow{f_1} Z_1 \rightarrow \dots \rightarrow Z_{s-1} \xrightarrow{f_s} Z_s = \tau^t H_0$$

in  $\text{ind } A$ . If each  $f_i$  is irreducible then  $s \geq n$  by the choice of  $X$  and  $Y$ . In this case we obtain  $Y \leq \tau Z_s = \tau^{t+1} H_0$  by a similar method as in the proof of Proposition 2.1. Thus we may assume that  $f_q$  is not a linear combination of compositions of irreducible maps but  $f_{q+1}, \dots, f_s$  are irreducible for some  $0 < q \leq s$ . Furthermore, we may assume  $s - q \geq n$ . It follows that  $Y \leq \tau^{t+1} H_0$  from the proof of Proposition 2.1. Inductively this proves our assertion. In particular,  $Y \leq \tau^m H_0 = X < Y$ , that is,  $Y$  is not directing. This contradicts Corollary 2.3. Q.E.D.

**Proposition 2.5.** *Let  $\mathcal{E}$  be a regular component of  $\Gamma_A$ . If  $\mathcal{E}$  contains some strongly directing  $A$ -module, then  $\mathcal{E}$  has only finitely many  $\tau$ -orbits and any indecomposable module belonging to  $\mathcal{E}$  is directing.*

*Proof.* If there is a DTr-periodic module in  $\mathcal{E}$ , then every indecomposable module in  $\mathcal{E}$  is DTr-periodic by [HPR]. This contradicts that  $\mathcal{E}$  contains a strongly directing  $A$ -module. So we can assume there exists no DTr-periodic module in  $\mathcal{E}$ . Let  $M \in \mathcal{E}$  be a strongly directing module. If there exists a nondirecting indecomposable module  $N \in \mathcal{E}$ , then we have a chain of nonzero and nonisomorphic morphisms in  $\text{ind } \mathcal{A}$ :

$$N = H_0 \xrightarrow{f_1} H_1 \rightarrow \dots \xrightarrow{f_t} H_t \xrightarrow{f_{t+1}} \dots \xrightarrow{f_{s-1}} H_{s-1} \xrightarrow{f_s} H_s \rightarrow \dots \xrightarrow{f_m} H_m = N,$$

where  $m > s > t$ . Because there exists no cycle of irreducible maps in  $\mathcal{E}$  by a theorem of Zhang [Z], we can assume  $f_1, \dots, f_t$  and  $f_s, \dots, f_m$  are irreducible maps but  $f_{t+1}$  and  $f_{s-1}$  are not compositions of irreducible maps; moreover, we can assume  $t$  and  $m - s$  large enough. Because the number of  $\tau$ -orbits in  $\mathcal{E}$  is finite by Corollary 2.4 and the form of  $\mathcal{E}$  is  $ZB$ ,  $B$  is a finite tree, by [Z]. So when  $t$  and  $m - s$  are large enough, we have  $H_s < M < H_t$ . Therefore  $M < N < M$ , a contradiction. This proves that every indecomposable module in  $\mathcal{E}$  is directing. Q.E.D.

**Corollary 2.6.** *The number of the  $\tau$ -orbits in  $\Gamma_A$  that contain strongly directing modules is finite.*

*Proof.* Let  $\mathcal{E}$  be a component of  $\Gamma_A$ . From Corollary 2.4 it is easy to see that  $\mathcal{E}$  admits only finitely many  $\tau$ -orbits that contain strongly directing modules. Thus, it is sufficient to consider the regular components of  $\Gamma_A$ . However, the regular component that contains strongly directing modules has only finitely

many  $\tau$ -orbits and consists entirely of directing modules by Proposition 2.5. It follows that there are only finitely many regular components of this kind from a result of Skowroński-Smalø [SS]. So the assertion is proved. Q.E.D.

**Theorem 2.7.** *Let  $A$  be an artin algebra. The  $\Gamma_A$  admits only finitely many  $\tau$ -orbits that contain directing modules.*

*Proof.* We want to prove this theorem by induction on the number of all pairwise orthogonal primitive idempotents. Put

$$\mathcal{M} = \{M_i, i \in I \mid M_i \text{ directing, } M_i \text{ and } M_j \text{ do not} \\ \text{belong to the same } \tau\text{-orbit for any } i \neq j\}.$$

If  $\mathcal{M}$  is an infinite set, then there are only finitely many strongly directing modules belonging to  $\mathcal{M}$ . Therefore there exists an infinite subset  $\mathcal{N}$  of  $\mathcal{M}$  and an indecomposable projective module  $P$  such that  $P$  is a successor of all modules in  $\mathcal{N}$ . Let  $e$  be the idempotent corresponding to  $P$ , that is,  $P \simeq Ae$ . Put  $A_0 = A/AeA$ . Then, for any predecessor  $X$  of some module in  $\mathcal{N}$ , the Auslander-Reiten sequence with the end term  $X$  is also an Auslander-Reiten sequence in  $\text{mod } A_0$ . In particular, the modules in  $\mathcal{N}$  belong to the distinct  $\tau_0$ -orbits, where  $\tau_0$  is the Auslander-Reiten operator in  $\text{mod } A_0$ . This is a contradiction to the induction hypothesis. Consequently  $\mathcal{M}$  must be a finite set. This finishes the proof of the theorem. Q.E.D.

*Remark.* After completing the revised form of this article, we received a preprint by A. Skowroński in which he independently obtained the same result as Theorem 2.7.

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#### REFERENCES

- [AR] M. Auslander and I. Reiten, *Representation theory of artin algebra III: Almost split sequences*, *Comm. Algebra* 3 (1975), 239–294.
- [B] K. Bongartz, *Tilted algebras*, *Lecture Notes in Math.*, vol. 903, Springer-Verlag, Berlin and New York, 1981, pp. 26–38.
- [BS] R. Bautista and S. Smalø, *Non existent cycles*, *Comm. Algebra* 11 (1983), 1755–1767.
- [HPR] D. Happel, U. Preiser, and C. M. Ringel, *Vinberg's characterization of Dynkin diagrams using subadditive functions with application to DTr-periodic modules*, *Lecture Notes in Math.*, vol. 832, Springer-Verlag, Berlin and New York, 1980, pp. 280–294.
- [R] C. M. Ringel, *Tame algebras and integral quadratic forms*, *Lecture Notes in Math.*, vol. 1099, Springer-Verlag, Berlin and New York, 1984.
- [SS] A. Skowroński and S. Smalø, *Directing modules*, *J. Algebra* (to appear).
- [Z] Y. Zhang, *The structure of stable components*, *Canad. J. Math.* 43 (1991), 652–672.

DEPARTMENT OF MATHEMATICS, SICHUAN UNIVERSITY, CHENGDU, 610064, SICHUAN, PEOPLES REPUBLIC OF CHINA

DEPARTMENT OF MATHEMATICS, BEIJING NORMAL UNIVERSITY, BEIJING, PEOPLES REPUBLIC OF CHINA