PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 118, Number 3, July 1993

THE STEENROD ALGEBRA ACTION ON GENERATORS OF RINGS OF INVARIANTS OF SUBGROUPS OF $\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$

NONDAS E. KECHAGIAS

(Communicated by Frederick R. Cohen)

ABSTRACT. We compute the action of the Steenrod algebra on rings of invariants of certain subgroups of $\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$, namely, the upper triangular and parabolic subgroups. As a consequence we get the action on the Dickson algebra for any prime p. The methods we use are elementary.

1. INTRODUCTION AND NOTATION

For the last two decades modular invariant theory has been playing an important role in Algebraic Topology. Hyunh [8], Mann [14], and Cooper [5] have used it to calculate the cohomology of Σ_{p^n} (the symmetric group on p^n letters) over $\mathbf{Z}/p\mathbf{Z}$. Because of the relation between Σ_{p^n} and (co-)homology operations, new definitions have been given in terms of modular invariant theory and a better understanding of the algebraic structure of these families of (co-)homology operations [8, 10].

Since any finite group G can be embedded in a symmetric group Σ_m, the cohomology of G is a finitely generated $H^*(\Sigma_m)$-module following Evens [7]. Moreover, since the Dickson algebra (defined below) "detects" [13, Definition 3.18, p. 54] $H^*(\Sigma_{p^n}, \mathbf{Z}/p\mathbf{Z})$, modular invariant theory plays an important role here [8, 13, 14].

The action of the Steenrod algebra on the Dickson algebra has been under investigation mainly because of its role in the realization of certain algebras as cohomology algebras (see [1, 4, 17, 18, 21]). The structure of this algebra as a module over the Steenrod algebra is still under investigation [2]. The rings of invariants that are the objects of study in this paper are intimately related to the construction of homology operations on certain kinds of spaces; for more details see [9, 10, 11].

In this paper we determine the Steenrod algebra action on generators of certain rings of invariants using elementary methods, for p an odd prime. The analogous results for $p = 2$ have been obtained by Hung [16]. This work is a revised form of a section of our Ph.D. thesis written under the supervision of H. E. A. Campbell; proofs concerning the calculation of the rings of invariants

Received by the editors August 30, 1991.
1991 *Mathematics Subject Classification*. Primary 55P99.
Key words and phrases. Modular invariants, Steenrod operations.

of parabolic subgroups can be found in [10]. Major changes were made in the notation, expression, and appearance all due to the referee whose time and patient is much appreciated. Moreover, his/her effort to teach us how to make it easier for the reader to follow was crucial for our first attempt at writing a paper. We would also like to thank John McCleary for his remarkable suggestions.

The notation we use follows Hyunh [8]. We recall the details here for the convenience of the reader.

The $\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$ action on

$$E \otimes P := E(x_1, \dots, x_n) \otimes P(y_1, \dots, y_n) \cong H^*(E_1 \times \cdots \times E_n, \mathbf{Z}/p\mathbf{Z})$$

is induced by the contragradient representation on $H'(E_1 \times \cdots \times E_n, \mathbf{Z}/p\mathbf{Z})$ (see also [13, p. 52]).

$A = E_1 \times \cdots \times E_n$ is the subgroup of Σ_{p^n} consisting of all translations on an n-dimensional vector space $V^n = \langle e_1, \dots, e_n \rangle$ over $\mathbf{Z}/p\mathbf{Z}$, where Σ_{p^n} acts by permuting its elements. Each E_i is the p-cyclic subgroup of Σ_p acting on V^n as translation by e_i. We have $\mathrm{Aut}(A) \cong \mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$ together with an n-dimensional representation $\rho : W_{\Sigma_{p^n}}(A) \to \mathrm{Aut}(A)$. Here $W_{\Sigma_{p^n}}$ denotes the Weyl group of A in Σ_{p^n}, that is, $W_{\Sigma_{p^n}}$ is the normalizer of A in Σ_{p^n} modulo the centralizer of A in Σ_{p^n}. Let the dual basis of V^n be $\{x_1, \dots, x_n\}$. Then $H^1(A) \cong (V^n)^*$ and the contragradient representation of ρ induces an action of $\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$ on $H^*(A, \mathbf{Z}/p\mathbf{Z})$, where this action is extended to the y_i's via the Bockstein monomorphism $\beta x_i = y_i$:

$$(g_{ij})x_s = \sum_{i=1}^{n} g_{is}x_i, \qquad (g_{ij})y_s = \sum_{i=1}^{n} g_{is}y_i \qquad \text{for } 1 \le s \le n.$$

Then $E \otimes P$ is a graded-commutative algebra over the Steenrod algebra with $|x_i| = 1$ and $|y_i| = 2$ for $1 \le i \le n$. By graded-commutative we mean $ab = (-1)^{|a||b|}ba$ for $a, b \in E \otimes P$.

Since the $\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$-action commutes with the Steenrod algebra A_p-action, $(E \otimes P)^G$ is an A_p-module, where G is any subgroup of $\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$.

We are mostly interested in the cases $G = U_n$, B_n, or $P_n(N)$. Here $P_n(N)$ denotes the parabolic subgroup of $\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$ associated to $N = (n_1, \dots, n_k)$ with $\sum n_i = n$ as follows:

$$P_n(N) = \begin{pmatrix} [n_1 \times n_1] & & & * \\ & [n_2 \times n_2] & & \\ & & \ddots & \\ 0 & & & [n_k \times n_k] \end{pmatrix}.$$

We have k blocks along the main diagonal, anything above, and 0 below, where any block $[n_i \times n_i]$ is an element of $\mathrm{GL}_{n_i}(\mathbf{Z}/p\mathbf{Z})$. Note that if $k = 1$, then $P_n(N) = \mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$, and if $n_i = i$, then $P_n(N)$ is denoted by B_n, a Borel subgroup of $\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$. Finally, let U_n be the subgroup of $\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$ consisting of matrices with 1's along the main diagonal, anything above, and zero below. It is well known that U_n is a p-Sylow subgroup of $\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$.

As was said before, the fundamental object of study of this paper is the Steenrod algebra action on Dickson's algebra, which in this paper will refer to $(E \otimes P)^{\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})}$. We recall that the object usually referred to as Dickson's algebra is $P^{\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})}$. The A_p-action on $P(y_1, \dots, y_n)^G$, for $G = U_n$

or $\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$, has been discussed by Campbell [3], Madsen [12], Madsen and Milgram [13], Singer [17], Smith and Switzer [18], and Wilkerson [20], and for $(E \otimes P)^{\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})}$ by Mann [14], May [15], and Hyunh [8]. Hung [16] classified the A_2-action on $P(y_1, \dots, y_n)^G$. First we recall some results concerning $(E \otimes P)^G$ from [8, 10]. Let L_n and $L_{n,i}$ denote, respectively, the following graded determinants (in the sense of Hyunh [8, p. 321]):

$$L_n = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1^p & \cdots & y_n^p \\ \vdots & & \\ y_1^{p^{n-1}} & \cdots & y_n^{p^{n-1}} \end{vmatrix} \quad \text{and} \quad L_{n,i} = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1^p & \cdots & y_n^p \\ \vdots & & \\ y_1^{p^n} & \cdots & y_n^{p^n} \end{vmatrix}.$$

Here the ith power is omitted from the second determinant, $0 \le i \le n-1$. Also let

$$Q_{n,i} = \frac{L_{n,i}}{L_n}.$$

Note. $L_n = \prod_{i=1}^n V_i$, where $V_i = \prod_{a_j \in \mathbf{Z}/p\mathbf{Z}} (a_1 y_1 + \cdots + a_{i-1} y_{i-1} + y_i)$ and $L_{n,0} = L_n^p$. Dickson's formula will be used in the sequel, so it is recalled here:

$$Q_{n,s} = Q_{n-1,s} V_n^{p-1} + Q_{n-1,s-1}^p.$$

The degrees of the previous invariants (of V_n, $\mathrm{SL}_n(\mathbf{Z}/p\mathbf{Z})$, and $\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$) are given by:

$$|V_i| = 2p^{i-1} \quad [2^{i-1}, \text{ if } p = 2],$$
$$|L_n| = 2(1 + \cdots + p^{n-1}) \quad [2^n - 1, \text{ if } p = 2],$$
$$|Q_{n,i}| = 2(p^n - p^i) \quad [2^n - 2^i, \text{ if } p = 2].$$

Next we discuss $\mathrm{SL}_n(\mathbf{Z}/p\mathbf{Z})$-invariants involving monomials consisting of x_i's and y_i's. We denote by $M_{n;s_1,\dots,s_m}$ the following determinant:

$$M_{n;s_1,\dots,s_m} = \frac{1}{m!} \begin{vmatrix} x_1 & \cdots & x_n \\ \vdots & & \\ x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \\ \vdots & & \\ y_1^{p^{n-1}} & \cdots & y_n^{p^{n-1}} \end{vmatrix}.$$

Here there are m rows of x_i's and the s_ith's powers are omitted, where $0 \le s_1 < \cdots < s_m \le n-1$. The degree is given by

$$|M_{n;s_1,\dots,s_m}| = m + 2((1 + \cdots + p^{n-1}) - (p^{s_1} + \cdots + p^{s_m})).$$

We state here the following theorems concerning the ring of invariants of U_n and $P_n(N)$.

Theorem [8, Theorem 5.6]. *The algebra* $(E \otimes P)^{U_n}$ *admits the decomposition*

$$P(V_1, \dots, V_n) \otimes \left(\bigoplus_{k=1}^{n} \bigoplus_{m=1} \bigoplus_{0 \le s_1 < \cdots < s_m = k-1} M_{k;s_1,\dots,s_m} \right).$$

Theorem [10, Theorem 3.15, p. 52]. *The algebra* $(E \otimes P)^{P_n(N)}$ *admits the decomposition*

$$P\left(Q_{v_j,i}, \left(\frac{L_{v_j}}{L_{v_{j-1}}}\right)^{p-1} \mid v_j = \sum_{t=1}^{j} n_t, \ 1 \le j \le k, \ v_{j-1} < i \le v_j - 1\right)$$
$$\otimes \left(\bigoplus_{j=1}^{k}\bigoplus_{m=1}^{v_j}\bigoplus_{l=1}^{n_j} \bigoplus_{0 \le s_1 < \cdots < s_m = v_j - l} M_{v_j; s_1, \ldots, s_m} L_{v_j}^{p-2}\right).$$

Here is Hyunh's explicit computation of Dickson's algebra.

Corollary [8, Theorem 4.17].

$$(E \otimes P)^{\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})} = P(Q_{n,i}, L_n^{p-1} \mid 1 \le i \le n-1)$$
$$\otimes \left(\bigoplus_{m=1}^{n} \bigoplus_{0 \le s_1 < \cdots < s_m} M_{n; s_1, \ldots, s_m} L_n^{p-2}\right).$$

Corollary [10, Theorem 3.13, p. 51].

$$(E \otimes P)^{B_n} = P(V_1^{p-1}, \ldots, V_n^{p-1}) \otimes \left(\bigoplus_{k=1}^{n}\bigoplus_{m=1}^{n} \bigoplus_{0 \le s_1 < \cdots < s_m = k-1} M_{k; s_1, \ldots, s_m} L_k^{p-2}\right).$$

Before stating our results we require some notation. We define $C(k) = C(k, s)$ to be the set of sequences $c = [c_0, \ldots, c_{s-1}]$ consisting of nonnegative integers which are solutions of the equation

$$k = \sum_{m=0}^{s-1} c_m (p^{s-1} + \cdots + p^{s-1-m}).$$

We note that $C(k)$ can be empty, that is, not all k admit such an expression. For example, $C(p^m)$ is empty if $m < s-1$. On the other hand, consider the numbers $p_m(s) = p_m = p^{s-1} + \cdots + p^{s-1-m}$ for $0 \le m \le s-1$ of the equation above: $C(p_m)$ has exactly one element, Δ_m, the sequence of which has all its entries 0 except for a 1 in the mth position from the left.

Suppose that we have a sequence $c \in C(k)$; we define $|c| = \sum c_m$ and we also define the *support* of c, denoted $\mathrm{supp}(c)$ to be the set of indices m for which $c_m \neq 0$. We further define $\gamma(c) = \gamma(c, s)$ as the coefficient mod p of $t^c = t_0^{c_0} \cdots t_{s-1}^{c_{s-1}}$ in the expression of $(t_0 + \cdots + t_{s-1})^{|c|}$, that is,

$$\gamma(c) = \binom{|c|}{c_0 \cdots c_{s-1}}.$$

As well, we define

$$Q_s^c = Q_{s,s-1}^{c_0} \cdots Q_{s,0}^{c_{s-1}}.$$

We are now in a position to define the polynomial

$$Q(k, s) = Q(k) = \sum_{c \in C(k)} (-1)^{|c|} \gamma(c) Q_s^c.$$

We make the following important conventions: first, when $k = 0$ that $Q(0, s) = 1$; and second, if $k > (p^s - 1)/(p-1)$ or $k < 0$, then $Q(k, s) = 0$; finally, if $C(k)$ is empty, we set $Q(k, s) = 0$. These conventions allow for the following succint statement of

Theorem 4.

$$P^k(L_i/L_s) = L_i/L_s\left(Q(k, s) + \sum_{t=0}^{i-1} Q_{i,i-1-t}Q(k - p_t(i), s)\right).$$

We note the special case $k < p^{i-1}$ when the second term is zero by convention since $k - p_t(i) < 0$ for all t, $0 \le t \le i-1$. Here we recall $p_t(i) = p^{i-1} + \cdots + p^{i-1-t}$. Furthermore, if $k = 0$ then we have $Q(0, s) = 1$ so that $P^0(L_i/L_s) = L_i/L_s$. This theorem will be proved in §2. It leads to

Theorem 7.

$$P^k(Q_{n,i}) = \sum_{s=0}^{i-1} Q_{n,s}Q(k - p_s(i), n) + Q_{n,i}Q(k, n).$$

In addition, we describe the action of the Steenrod algebra on the 'exterior' invariants at the end of this paper.

All of the theorems in this paper have analogues at the prime 2. Put briefly, one ignores the exterior algebras (set $x_i = 0$) and regrades $P(y_1, \ldots, y_n)$ by taking the degree of y_i to be 1 when $p = 2$. Using these conventions, we restate our results as they apply to the prime 2 in Corollaries 6 and 10.

2. Computations

To determine the A_p-action on the generators of the previous rings of invariants we use the following lemma due to Steenrod, the Cartan formula, and properties of the generators.

Lemma 1 (Steenrod). *Let x and y be* mod p *cohomology classes in any space such that $|x| = 1$ and $|y| = 2$. Then*:

$$P^i x = 0, \quad \textit{unless } i = 0.$$
$$P^i y^k = \binom{k}{i} y^{k+i(p-1)}.$$

In particular,

$$P^i y^{p^k} = \begin{cases} y^{p^k} & \textit{if } i = 0, \\ y^{p^{k+1}} & \textit{if } i = p^k, \\ 0 & \textit{otherwise.} \end{cases}$$

We begin our work with the generators of the polynomial algebras, namely:

$$\frac{L_n}{L_i} = \prod_{j=i+1}^{n} V_j, \quad \frac{L_n}{L_{n-1}} = V_n, \quad Q_{n,i}, \quad \text{and} \quad V_n^{p-1}.$$

Using the fact that any monomial of $L_{n,i}$ is of the form $y_1^{p^{i_1}} \cdots y_n^{p^{i_n}}$, the Cartan formula, and Lemma 1, we obtain the following.

Lemma 2.

(a) $P^j(L_n) = \begin{cases} L_{n,k} & \text{if } j = p_{n-1-k} = p^{n-1} + \cdots + p^k, \\ 0 & \text{otherwise.} \end{cases}$

(b) $P^j L_{n,i} = \begin{cases} L_{n,k} & \text{if } j = p_{i-1-k} = p^{i-1} + \cdots + p^k, \\ L_{n,i}Q^p_{n,n-1} - Q^p_{n,i-1}L_{n,k} & \text{if } j = p_{n-k}(n+1) = p^n + \cdots + p^k \\ & \text{and } i+1 \le k \le n, \\ L_{n,l}Q^p_{n,k-1} - Q^p_{n,l-1}L_{n,k} & \text{if } j = p_{n-k}(n+1) + p_{i-1-l}(i) \\ & \text{and } i+1 \le k \le n, \\ 0 & \text{otherwise.} \end{cases}$

Note. We recall that $L_{n,0} = L^p_n$.

In order to prove Lemma 2(b), we will need to derive the following formula:

$$L_{n;i,k} = L_{n,i}Q^p_{n,k-1} - Q^p_{n,i-1}L_{n,k}.$$

Here $L_{n;i,k}$ denotes the following determinant where the ith and kth powers are omitted:

$$L_{n;i,k} = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1^p & \cdots & y_n^p \\ \vdots & & \\ y_1^{p^{n+1}} & \cdots & y_n^{p^{n+1}} \end{vmatrix}.$$

Moreover, we define $L_{n;i,k,m}$ in a similar form. The following sublemmas are used in the derivation of this formula.

Sublemma 2. (a) $L_{n-2;l,i}L_{n-2;k} + L_{n-2;i,k}L_{n-2;l} = L_{n-2;i}L_{n-2;l,k}$ *for* $0 \le l < i < k < n-1$ *and* $n > 3$.

(b) $L_{n-2;l,i,n-1}L_{n-2;k} + L_{n-2;i,k,n-1}L_{n-2;l} = L_{n-2;i}L_{n-2;l,k,n-1}$ *for* $0 \le l < i < k < n-1$ *and* $n > 3$.

Proof. The proof of Sublemma 2(a) is by induction on n (expand the determinants along the last row). Sublemma 2(b) is proved by applying $P^{p^{n-1}}$ to the equation of Sublemma 2(a).

Suppose we want to define the action on A and that A is related with B and C by $AB = C$, where the action has been determined on both B and C. The P^kA is given with respect to P^mB and P^lC for various m and l which depend on k using the Cartan formula. For example, in the proof of Lemma 3 given below, we take $A = L_i/L_s$.

We use the following lemma in the proof of Lemma 3, and it is also required in the proof of Theorems 4 and 7, which have been omitted here.

Lemma.

$$Q(k, s) = -\sum_{m=0}^{s-1}\Big(\sum_{c \in C(k-p_m, s)} Q(k - p_m, s)\Big)Q_{s,s-1-m}.$$

Proof. We note that if $m \in \operatorname{supp}(c)$ for $c \in C(k, s) = C(k)$ then $c(m) = c - \Delta_m \in C(k - p_m)$ and so $|c| = |c(m)| + 1$. Furthermore, $Q^c_s = Q^{c(m)}_s Q_{s,s-1-m}$.

Also, if $d \in C(k - p_m)$ then $d + \Delta_m \in C(k)$. We have only to show now that

$$\gamma(c) = \sum_{l \in \mathrm{supp}(c)} \gamma(c(l)).$$

We consider the equation

$$\begin{aligned}(t_0 + \cdots + t_{s-1})^{|c|} &= (t_0 + \cdots + t_{s-1})(t_0 + \cdots + t_{s-1})^{|c|-1} \\ &= (t_0 + \cdots + t_{s-1}) \left(\sum_{l \in \mathrm{supp}(c)} \gamma(c(l)) t^{c(l)}) + \text{other terms} \right) \\ &= \sum_{l \in \mathrm{supp}(c)} (\gamma(c(l)) t^{c(l)}) t_{s-1-l} + \text{other terms} \\ &= \gamma(c) t^c + \text{other terms},\end{aligned}$$

as required.

Lemma 3. *For* $0 \le k < p^{i-1}$ *we have*

$$P^k(L_i/L_s) = (L_i/L_s) Q(k, s).$$

We recall that $Q(k, s)$ *is defined just prior to the statement of Theorem* 4 *in the introduction.*

Proof. The proof is by induction on k. We use the abbreviation $p_m = p^{s-1} + \cdots + p^{s-1-m}$. We could start with $k = 0$, but we learn more by starting with $0 < k \le p^{s-1} + \cdots + p + 1 = p_{s-1}$. We consider the equation

$$P^k(AL_s) = P^k(L_i).$$

The right-hand side of this equation is zero because $k < p^{i-1}$ (see Lemma 2(a)). Applying the Cartan formula and Lemma 2(a) to the left-hand side, we obtain

$$P^k(A)L_s + AP^k(L_s) = P^k(A)L_s + AL_{s,s-1-m}$$

if and only if $k = p_m$ for some m, $0 \le m \le s - 1$; otherwise, $P^k(A) = 0$. That is, we have $P^k(A) = -AQ_{s,s-1-m}$ if $k = p_m$ and $= 0$ otherwise. On the other hand, if $k = p_m$ we have $C(k) = \{\Delta_m\}$ and the formula of the lemma gives also $P^{p_m}(A) = -AQ_{s,s-1-m}$ as reqired. Otherwise, in this range, $C(k)$ is empty, and the formula gives zero by convention.

So we may assume the formula of the lemma holds for those l with $p_{s-1} \le l < k \le p^{i-1}$. We once again consider the equation

$$P^k(AL_s) = P^k(L_i) = 0,$$

which may be rewritten, using the Cartan formula, Lemma 2(a), and induction as

$$\begin{aligned}-P^k(A) &= \sum_{l=1}^{k} P^{k-l}(A) P^l(L_s)/L_s = \sum_{m=0}^{s-1} P^{k-p_m}(A) L_{s,s-1-m}/L_s \\ &= \sum_{m=0}^{s-1} \left(\sum_{c \in C(k-p_m)} Q(k - p_m, s) \right) Q_{s,s-1-m}.\end{aligned}$$

The proof now follows immediately from the lemma above.

The proof of the remainder of Theorem 4 is a similar induction and so we omit it.

Corollary 5.

$$P^k(V_i) = V_i(Q(k\,,\,i-1) + Q_{i,i-1}(Q(k-p_0(i)\,,\,i-1))).$$

Proof. We take $s = i-1$ in Theorem 4. We note that the second term is zero by convention unless $k = p^{i-1}$ when the formula yields

$$V_i(-Q^p_{i-1,i-2} + Q_{i,i-1})$$

which equals V_i^p by Dickson's formula, quoted in the introduction.

Corollary 6 [13]. *Let* $p = 2$. *Then*

$$Sq^k(V_i) = \begin{cases} V_iQ_{i-1,i-2-t} & \text{if } k = 2^{i-2} + \cdots + 2^{i-1-t}, \\ V_i^2 & \text{if } k = 2^{i-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Theorem 7. We sketch here the proof of this theorem. We again use the previous method, namely, we observe that $Q_{n,i}L_n = L_{n,i}$. So we apply P^k to both sides of this equation. Using the Cartan formula and Lemma 2 and proceeding as above we obtain the desired result.

Corollary 8 [19].

$$P^{p^j}(Q_{n,i}) = \begin{cases} Q_{n,i-1} & \text{if } j = i-1, \\ -Q_{n,i}Q_{n,n-1} & \text{if } j = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. According to Theorem 7, we must find the j's such that either of the sets $C(p^j\,,\,n)$ or $C(p^j - p_s(i)\,,\,n)$ is not empty. Considering the restrictions for j: $p^{i-1} \le p^j \le p^{n-1}$, we conclude that the first set contains only one sequence when $j = n-1$ and the second one when $j = i-1$, the zero sequence (a nonzero solution requires $j > n-1$, which contradicts the restriction).

Corollary 9 [15]. *Let* $p = 2$. *Then*

$$Sq^k(Q_{n,i}) = \begin{cases} Q_{n,n-1-m}Q_{n,s} & \text{if } k = 2^n - 2^{n-1-m} + 2^i - 2^s \\ & \text{and } 0 \le m < n-1-i, \\ Q_{n,s} & \text{if } k = 2^i - 2^s, \\ Q^2_{n,i} & \text{if } k = 2^n - 2^i, \\ 0 & \text{otherwise.} \end{cases}$$

As a final application we compute $P^kV_n^{p-1}$. We recall that V_n^{p-1} is a generator of $(P(y_1\,,\,\ldots\,,\,y_n))^{B_n}$ where B_n is the group of upper triangular matrices. We use $Q_{n-1,0}V_n^{p-1} = Q_{n,0}$, the Cartan formula, and Theorem 7 to get

Theorem 10.

$$P^k(V_n^{p-1}) = \begin{cases} \frac{1}{Q_{n-1,0}}\Big(P^k(Q_{n,0}) - V^{p-1}(P^k(Q_{n-1,0}) \\ \qquad\qquad + \sum_{m=0}^{n-2} Q_{n-1,n-2-m}P^{k-p_m}(Q_{n-1,0}))\Big), \\ 0 \quad \text{if } C(k) = \varnothing. \end{cases}$$

Proof. The proof depends on the following observations and the Cartan formula:

(a) $P^{p_m}(V_n^{p-1}) = V_n^{p-1}Q_{n-1,n-2-m}$ for $p_m(n-1) = p_m = p^{n-2} + \cdots + p^{n-2-m}$.

(b) $P^{lp_m}(V_n^{p-1}) = 0$ if $l \neq 1$ and $lp_m < p^{n-1}$.
(c) $P^{p_m+p_t}(V_n^{p-1}) = 0$ if $m \neq t$ and $p_m + p_t < p^{n-1}$.

Corollary 11.

$$P^{p^k}(V_n^{p-1}) = \begin{cases} -V_n^{p-1}Q_{n-1,n-2} & \text{if } k = n-2, \\ (V_n^{p-1}(V_n^{p-1} + 2Q_{n-1,n-2}^p)) & \text{if } k = n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Now we compute the A_p-action on certain generators of $(E \otimes P)^G$ involving $M_{n;s_1,\dots,s_k}$. Note that any monomial in $M_{n;s_1,\dots,s_k}$ is of the form $x_{i_1}\cdots x_{i_k} y_{j_1}^{p^{t_1}} \times y_{j_2}^{p^{t_2}} \cdots y_{j_{n-k}}^{p^{t_{n-k}}}$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, and $\{j_1, \dots, j_{n-k}\}$ is the complement of $\{i_1, \dots, i_k\}$ in $\{1, \dots, n\}$. Hence the A_p-action on $M_{n;s_1,\dots,s_k}$ is completely determined by Lemma 1 and the Cartan formula:

Lemma 12.

$$\text{(a) } \beta M_{n;s_1,\dots,s_k} = \begin{cases} (-1)^{k-1} M_{n;s_2,\dots,s_k} & \text{if } s_1 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{(b) } P^{p^k}(M_{n;s}) = \begin{cases} M_{n;s-1} & \text{if } k = s-1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{(c) } P^{p^{n-1}}(M_{n;s}) = \begin{cases} M_{n;s}Q_{n,n-1} - M_{n;n-1}Q_{n,s} & \text{if } s \neq n-1, \\ 0 & \text{if } s = n-1. \end{cases}$$

Proof. (a) and (b) are obvious. For (c) we note that the determinant given by $M_{n;s}$, where its last row has been replaced by $y_1^{p^n} \cdots y_n^{p^n}$, can be expressed by $M_{n;s}Q_{n,n-1} - M_{n;n-1}Q_{n,s}$ (see [10, Lemma 2.20, p. 69]).

Corollary 13.

$$\text{(a) } P^{p^i}(M_{n;s_1,\dots,s_k}) = \begin{cases} M_{n;s_1,\dots,s_j-1,\dots,s_k} & \text{if } i = s_j - 1 \text{ and } s_{j-1} \neq s_j - 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{(b) } P^{p^{n-1}}(M_{n;s_1,\dots,s_k}) = \begin{cases} M_{n;s_1,\dots,s_k}Q_{n,n-1} \\ \quad + \sum_{i=1}^k (-1)^i M_{n;s_1,\dots,\widehat{s_i},\dots,s_k,n-1}Q_{n,s_i} \\ \qquad \text{if } s_k < n-1, \\ 0 \quad \text{if } s_k \geq n-1. \end{cases}$$

Here $M_{n;s_1,\dots,\widehat{s_i},\dots,s_k,n-1}$ *means the* s_i *index is missing.*

Finally,

Theorem 14. (a) *The value of* P^{p^i} *on the invariant* $M_{n;s_1,\dots,s_k}L_n^{p-2}$ *is*

$$\begin{cases} M_{n;s_1,\dots,s_j-1,\dots,s_k}L_n^{p-2} & \text{if } i = s_j - 1 \text{ and } s_{j-1} \neq s_j - 1, \\ 0 & \text{if } i = s_{j-1} = s_j - 1 \text{ or } i - 1 \neq s_j, n, \\ -L_n^{p-2}\left(M_{n;s_1,\dots,s_k}Q_{n,n-1} + \sum_{i=1}^k (-1)^i M_{n;s_1,\dots,\widehat{s_i},\dots,s_k n-1}Q_{n,s_i}\right) & \\ \qquad\qquad \text{if } i = n-1 \text{ and } s_k < n-1, \\ (p-2)M_{n;s_1,\dots,s_k}L_n^{p-2}Q_{n,n-1}, & \text{if } i = n-1 \text{ and } s_k = n-1. \end{cases}$$

(b) *The value of* β *on the same invariant above,* $M_{n;s_1,\dots,s_k}L_n^{p-2}$, *is*

$$\begin{cases} (-1)^{k-1}M_{n;s_2,\dots,s_k}L_n^{p-2} & \text{if } s_1 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Using the methods above the reader may discover the formulas for the action of P^k for any k on the invariants of Theorem 14.

Bibliography

1. J. F. Adams and C. Wilkerson, *Finite H-spaces and algebras over the Steenrod algebra*, Ann. of Math. (2) **111** (1980), 95–143.
3. A. Adem, J. Maginnis, and R. J. Milgram, *Symmetric invariants and cohomology of groups*, Ann. of Math. (2) **287** (1990), 391–411.
3. H. E. A. Campbell, *Upper triangular invariants*, Canad. Math. Bull **28** (1985), 243–248.
4. H. E. A. Campbell and P. S. Selick, *Polynomial algebras over the Steenrod algebra*, Comment. Math. Helv. **64** (1990), 171–180.
5. B. Cooper, *Cohomology of symmetric groups*, mimeo, Augsberg College, 1975.
6. L. E. Dickson, *A fundamental system of invariants of the general modular linear group with a solution of the form problem*, Trans. Amer. Math. Soc. **12** (1911), 75–98.
7. L. Evens, *The cohomology ring of a finite group*, Trans. Amer. Math. Soc. **101** (1961), 224–239.
8. Hyunh Múi, *Modular invariant theory and cohomology algebras of symmetric groups*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **22** (1975), 319–369.
9. ——, *Homology operations derived from modular coinvariants*, Lecture Notes in Math., vol. 172, Springer-Verlag, New York, 1985, pp. 85–115.
10. N. E. Kechagias, Ph.D. thesis, Queen's University, Kingston, Ontario, 1990.
11. ——, *Mod odd modular coinvariants, homology operations, and limit spaces*, preprint.
12. I. Madsen, *On the action of the Dyer-Lashof algebra in* $H_*(F)$, Pacific J. Math. **60** (1975), 235–275.
13. I. Madsen and R. J. Milgram, *The classifying spaces for surgery and cobordism of manifolds*, Ann. of Math. (2) **92** (1979).
14. B. N. Mann, *The cohomology of the symmetric groups*, Trans. Amer. Math. Soc. **242** (1978), 157–184.
15. J. P. May, *The homology of* E^∞*-spaces*, Lecture Notes in Math., vol. 533 , Springer-Verlag, New York, 1975.
16. Nguyén H. V. Hung, *The action of the Steenrod squares on the modular invariants of linear groups*, Proc. Amer. Math. Soc. **113** (1991), 1097–1104.
17. W. M. Singer, *On the construction of certain algebras over the Steenrod algebra*, J. Pure Appl. Algebra **11** (1977), 53–59.
18. L. Smith and R. Switzer, *Realizability and non-realizability of Dickson algebras as cohomology rings*, Proc. Amer. Math. Soc. **89** (1983), 303–313.
19. N. E. Steenrod and D. B. A. Epstein, *Cohomology operations*, Ann. of Math. Stud., no. 50, Princeton Univ. Press, Princeton, NJ, 1962.
20. C. Wilkerson, *A primer on the Dickson invariants*, Contemp. Math., vol. 19, Amer. Math. Soc., Providence, RI, 1983, pp. 421–434.
21. R. M. W. Wood, *Steenrod squares of polynomials*, London Math. Soc. Lecture Note Ser., vol. 139, Cambridge Univ. Press, Cambridge and New York, 1989, pp. 173–177.

Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, Canada K7L 3N6

Current address: Department of Mathematics, University of Crete, Iraklion, Crete, Greece