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AN OSCILLATION CRITERION FOR A FORCED SECOND-ORDER LINEAR DIFFERENTIAL EQUATION

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ABSTRACT. The paper is devoted to an oscillation theorem for the second-order forced linear differential equation of the form $(p(t)x')' + q(t)x = g(t)$. The sign of the coefficient q is not definite, and the function g is not necessarily the second derivative of an oscillatory function. The question raised by J. Wong in *Second order nonlinear forced oscillations* (SIAM J. Math. Anal. **19** (1988), 667–675) is answered. A region of oscillation of Mathieu's equation is specified.

1. INTRODUCTION

We are concerned here with an oscillation criterion for a second-order forced linear differential equation of the form

$$(p(t)x')' + q(t)x = g(t), \qquad t \in [0, \infty[, \tag{1.1}$$

where p, q, g are continuous functions in $[0, \infty[$, $p > 0$, and $x, px' \in C^{(1)}(0, \infty)$. We assume that g is an oscillatory function and that q is of arbitrary sign.

The widely used method, suggested by Kartsatos [3, 4] (see also [9]) in his study of forced oscillation, imposes a restriction on the function g. Namely, g must be the second derivative of an oscillatory function. Our study is free of this restriction. We assume only that g is oscillatory. Also, we do not impose any restrictions on the sign of the coefficient q.

Hartman, Hille, Leighton, Nehari, Wintner, and others (cf. [7] and the literature cited there) found oscillation criteria for second-order linear homogeneous differential equations. The methods used involve the integration (or the average) of the function q in the whole interval $[0, \infty[$ and do not give any information about the distribution of the zeros of solutions.

The method used here to study the oscillation behaviour of equation (1.1) depends on a comparison theorem of Sturm's type due to Leighton [5]. It gives a criterion depending only on the behaviour of q in certain intervals. Outside these intervals the behaviour of q is irrelevant. Also, information about the distribution of the zeros of solutions is obtained.

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Using this method, we are able to investigate a number of linear differential equations, forced and unforced. For example, the following equations, the first of which is Mathieu's equation [8], are oscillatory:

$$(1)\quad x'' + (a + b\cos 2t)x = 0, \qquad a + \tfrac{1}{2}|b| \geq 1, \ a, b \in \mathbb{R};$$
$$(2)\quad x'' + (1 + 2\cos t)x = \sin t, \qquad t \in [0, \infty[.$$

It is remarkable that, as a special case of Mathieu's equation, if $a = -1$ and $|b| \geq 4$, then we have $q(t) = -1 + b\cos 2t$ and therefore

$$\int_0^\infty q(t)\,dt = -\infty.$$

Hence, none of the known criteria can be applied to this case. Again, in Mathieu's equation, if we set $a = 0$ and $|b| \geq 2$, we get $q(t) = b\cos 2t$ and the integral $\int_0^\infty q(t)\,dt$ does not exist. The known criteria cannot be applied here either.

Wong [9] noticed that Howard's results [2] do not apply to many examples with arbitrary q. Wong considers the answer of whether (2) is oscillatory as a major step forward in the study of forced oscillations.

The main results of the paper are formulated in Theorems 3.1 and 3.2 of §3. Section 4 contains some examples of forced and unforced oscillations, including Mathieu's equation (1) and Wong's equation (2).

2. Preliminaries

For the convenience of the reader we start our study with the following definition and lemma (see [1, 6–7]).

Definition. Let

$$\mathscr{D} = \{u \in C^{(1)}[\alpha, \beta] : u(\alpha) = u(\beta) = 0, \alpha, \beta \in \mathbb{R}, \alpha < \beta\},$$

and let

$$(Lx)(t) = (p(t)x')' + q(t)x$$

be a linear differential operator with domain of definition $\mathscr{D}_L = \{u : u, pu' \in C^{(1)}[\alpha, \beta]\}$. A function $u \in \mathscr{D}_L$ satisfying the inequality $Lu \leq 0$ ($Lu \geq 0$) on the interval $[\alpha, \beta]$ is called an L-subsolution (L-supersolution).

Consider the quadratic functional j defined by

$$j[u] = \int_\alpha^\beta \tilde{p}u'^2 - \tilde{q}u^2, \qquad u \in \mathscr{D},$$

which corresponds to the linear differential equation

$$(ly)(t) = (\tilde{p}(t)y')' + \tilde{q}(t)y = 0, \qquad t \in [0, \infty[,$$

where $\tilde{p}$ and $\tilde{q}$ are continuous functions on $[0, \infty[$, $\tilde{p} > 0$, and

$$y \in \mathscr{D}_l = \{u : u, \tilde{p}u' \in C^{(1)}[\alpha, \beta]\}.$$

One can easily prove that

$$\int_\alpha^\beta ulu + j[u] = \tilde{p}(t)u(t)u'(t)|_\alpha^\beta.$$

Defining on $\mathscr{D}$, in addition to j, the quadratic functional

$$J[u] = \int_\alpha^\beta pu'^2 - qu^2,$$

and setting

$$V[u] = j[u] - J[u], \qquad u \in \mathscr{D},$$

we have the following

Lemma. *If there exists a nontrivial real solution u of $lu = 0$ in $]\alpha, \beta[$, $u \in \mathscr{D}$, such that $V[u] \geq 0$, then every positive L-subsolution (negative L-supersolution) has a zero in $]\alpha, \beta[$ unless it is a constant multiple of u.*

Proof. Since $lu = 0$, $u \in \mathscr{D}$, this implies that $j[u] = 0$. Therefore, $J[u] \leq 0$ and in both cases $uLu \leq 0$. Hence the lemma follows from [1, Theorem 8, p. 11].

3. Forced oscillation criterion

In this section, we state and prove the main theorem of the paper on the oscillation of forced (and unforced) second-order linear differential equations, with alternating coefficient. In the next section we examine some examples.

Theorem 1. *Let there exist two positive increasing divergent sequences $\{a_n^+\}$, $\{a_n^-\}$ and two sequences of positive numbers $\{c_n^+\}$, $\{c_n^-\}$ such that*

$$V_n^\pm = \int_{a_n^\pm}^{a_n^\pm + \pi/\sqrt{c_n^\pm}} (c_n^\pm[1 - p(t)]\cos^2\{\sqrt{c_n^\pm}(t - a_n^\pm)\} + [q(t) - c_n^\pm]\sin^2\{\sqrt{c_n^\pm}(t - a_n^\pm)\})\, dt \geq 0 \tag{3.1}$$

for every $n \in \mathbb{N}$. Assume that a function g satisfies

$$g(t)\begin{cases} \geq 0, & t \in [a_n^+, a_n^+ + \pi/\sqrt{c_n^+}], \\ \leq 0, & t \in [a_n^-, a_n^- + \pi/\sqrt{c_n^-}], \end{cases}$$

for every $n \in \mathbb{N}$. Then the linear forced equation

$$(Ly)(t) = (p(t)y')' + q(t)y(t) = g(t), \qquad t \in [0, \infty[,$$

is oscillatory.

Proof. If we suppose, to the contrary, that $Ly = g$ has an eventually positive solution, then there exists $n_0 \in \mathbb{N}$ such that $y(t) > 0 \quad \forall t \geq n_0$. This solution in the intervals $[a_n^-, a_n^- + \pi/\sqrt{c_-^n}]$ satisfies $Ly \leq 0$.

Consider the linear homogeneous differential equation

$$x''(t) + c_n^- x(t) = 0, \qquad t \in [a_n^-, a_n^- + \pi/\sqrt{c_n^-}], \quad n \geq n_0.$$

This equation has the solution $u(t) = \sin\{\sqrt{c_n^-}(t - a_n^-)\}$, which has two consecutive zeros at $t = a_n^-$ and at $t = a_n^- + \pi/\sqrt{c_n^-}$. Therefore, we have a positive L-subsolution that satisfies $V_n^- \geq 0$, $n \geq n_0$. By the above lemma y has a zero in $]a_n^-, a_n^- + \pi/\sqrt{c_n^-}[$ unless y is a constant multiple of $\sin\{\sqrt{c_n^-}(t - a_n^-)\}$. Both cases lead to a contradiction.

Next, suppose that the solution is eventually negative. We use the fact that $Ly \geq 0$ in $[a_n^+, a_n^+ + \pi/\sqrt{c_n^+}]$ and $y < 0$ for all n greater than or equal to some n_0 to get a contradiction.

In the case of unforced equations, i.e., $g \equiv 0$, we have the following theorem, which is a special case of Theorem 1.

Theorem 2. *If there exists an increasing divergent sequence of positive numbers* $\{a_n\}$ *and a sequence of positive numbers* $\{c_n\}$ *such that*

$$\begin{aligned} V_n = \int_{a_n}^{a_n+\pi/\sqrt{c_n}} &(c_n[1-p(t)]\cos^2\{\sqrt{c_n}(t-a_n)\} \\ &+ [q(t)-c_n]\sin^2\{\sqrt{c_n}(t-a_n)\})\,dt \geq 0, \end{aligned} \tag{3.2}$$

then the equation $[p(t)x']' + q(t)x = 0$ *is oscillatory.*

4. The oscillation of Mathieu's equation and others

In this section we examine Mathieu's equation. Two more examples are given. The first is a generalization of the question raised by Wong in [9], while the second is rather illustrative.

Example 1. (a) Let $a, b \in \mathbb{R}$ and $a + \frac{1}{2}|b| \geq 1$. Then Mathieu's equation

$$x''(t) + (a + b\cos 2t)x = 0, \qquad t \in [0, \infty[,$$

is oscillatory. In fact, every solution of the equation has a zero in every interval $[(n-\frac{1}{2})\pi, (n+\frac{1}{2})\pi]$ if $b \geq 0$ and in $[(n-1)\pi, n\pi]$ if $b < 0$, $n \in \mathbb{N}$.

(b) Let $a, b \in \mathbb{R}$ and $a > 0$. Then Mathieu's equation

$$x''(t) + (a + b\cos 2t)x = 0, \qquad t \in [0, \infty[,$$

is oscillatory. Moreover, there exists a natural number n_0 such that every solution of the equation has a zero in every interval $[(n-1)\frac{\pi}{2}, (n+n_0-1)\frac{\pi}{2}]$ of length $\frac{\pi}{2}n_0$.

Proof. (a) In (3.2), setting $p(t) = 1$, $q(t) = a + b\cos 2t$, $c_n \equiv 1$, $a_n = (n-\frac{1}{2})\pi$ if $b \geq 0$, or $a_n = (n-1)\pi$ if $b < 0$, $n \in \mathbb{N}$, we get

$$V_n = \frac{\pi}{2}\left(a + \frac{1}{2}\mid b\mid -1\right) \geq 0.$$

Note that, in case $a = -1$ or $a = 0$ Mathieu's equation is oscillatory if $|b| \geq 4$ and $|b| \geq 2$ respectively. However, in the first case $\int_0^\infty q(t)\,dt = -\infty$ and in the second $\int_0^\infty q(t)\,dt$ does not exist. Therefore none of the known criteria can be applied to these cases.

(b) Setting $p(t) = 1$, $q(t) = a+b\cos 2t$, $c_n \equiv (2/n)^2$, $a_n = (n-1)\frac{\pi}{2}$, $n \in \mathbb{N}$, in (3.2), we get

$$V_n = \frac{n\pi}{4}\left(a - \frac{4}{n^2}\right).$$

Since $a > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $a - 1/n^2 \geq 0$. Now choosing $c_n = c_{n_0}$ for all $n \geq n_0$, we get $V_n \geq 0$ $\forall n \geq n_0$.

Example 2. Let α, β, γ, $\delta \in \mathbb{R}$ and $\alpha < 1$. Then the equation

$$[(1 - \alpha \cos t)x']' + (1 + \beta \cos t)x = \gamma t^{\delta} \sin t, \qquad t \in]0, \infty[,$$

is oscillatory. In particular, for $\alpha = 0$, $\beta = 2$, $\gamma = 1$, $\delta = 0$, we have that

$$x'' + (1 + 2\cos t)x = \sin t, \qquad t \in [0, \infty[,$$

is oscillatory.

Proof. In (3.1) set $p(t) = 1 - \alpha \cos t$, $q(t) = 1 + \beta \cos t$, and $c_n \equiv 1$. Then

$$V_n^{\pm} = -\tfrac{2}{3}(\alpha + 2\beta) \sin a_n^{\pm}.$$

If $\gamma \geq 0$, choose $a_n^+ = 2(n-1)\pi$ and $a_n^- = (2n-1)\pi$, $n \in \mathbb{N}$. Then $V_n = 0$. In the intervals $[a_n^+, a_n^+ + \pi]$ the function $g(t) = \gamma t^{\delta} \sin t \geq 0$, while in $[a_n^-, a_n^- + \pi]$ the function $g \leq 0$. In case $\gamma < 0$, the same holds if we interchange a_n^- and a_n^+.

Example 3. As an illustrative example, we consider the equation

$$[(\tfrac{3}{2} - \sin t)x']' + 16(\sin t - \tfrac{1}{4})x = \gamma t^{\delta} \cos t, \qquad \gamma, \delta \in \mathbb{R}, \ t > 0.$$

This equation is oscillatory and every solution has a zero in every one of the following intervals: $[2(n-1)\pi, (2n - \frac{3}{2})\pi]$ and $[(2n - \frac{3}{2})\pi, (2n-1)\pi]$, $n \in \mathbb{N}$.

Proof. Setting $p(t) = \frac{3}{2} - \sin t$, $q(t) = 16(\sin t - \frac{1}{4})$, $a_n^+ = 2(n-1)\pi$, $a_n^- = (2n - \frac{3}{2})\pi$, and $c_n^{\pm} = 4$, one can easily obtain that $V_n^{\pm} = 10[1 - \frac{\pi}{4}] + \frac{2}{5}$. If $\gamma \geq 0$, the function $g(t) = \gamma t^{\delta} \cos t \geq 0$ in $[a_n^+, a_n^+ + \frac{\pi}{2}]$ and $g(t) \leq 0$ in $[a_n^-, a_n^- + \frac{\pi}{2}]$. If $\gamma < 0$, we apply the same technique used in Example 2.

In Examples 2 and 3, we can replace γt^{δ} with a more general function $f(t)$. For example, $f(t)$ may be any arbitrary piecewise continuous function of definite sign in $]0, \infty[$. In this case the assertions of Examples 2 and 3 hold.

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