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INTEGER-VALUED POLYNOMIALS, PRÜFER DOMAINS, AND LOCALIZATION

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ABSTRACT. Let A be an integral domain with quotient field K and let $\mathrm{Int}(A)$ be the ring of integer-valued polynomials on A: $\{P \in K[X] | P(A) \subset A\}$. We study the rings A such that $\mathrm{Int}(A)$ is a Prüfer domain; we know that A must be an almost Dedekind domain with finite residue fields. First we state necessary conditions, which allow us to prove a negative answer to a question of Gilmer. On the other hand, it is enough that $\mathrm{Int}(A)$ behaves well under localization; i.e., for each maximal ideal $\mathfrak{m}$ of A, $\mathrm{Int}(A)_{\mathfrak{m}}$ is the ring $\mathrm{Int}(A_{\mathfrak{m}})$ of integer-valued polynomials on $A_{\mathfrak{m}}$. Thus we characterize this latter condition: it is equivalent to an "immediate subextension property" of the domain A. Finally, by considering domains A with the immediate subextension property that are obtained as the integral closure of a Dedekind domain in an algebraic extension of its quotient field, we construct several examples such that $\mathrm{Int}(A)$ is Prüfer.

INTRODUCTION

Throughout this paper A is assumed to be a domain with quotient field K, and $\mathrm{Int}(A)$ denotes the ring of integer-valued polynomials on A:

$$\mathrm{Int}(A) = \{P \in K[X] | P(A) \subset A\}.$$

The case where A is a ring of integers of an algebraic number field K was first considered by Pólya [13] and Ostrowski [12]. In this case we know that $\mathrm{Int}(A)$ is a non-Noetherian Prüfer domain [1, 4]. More generally, if A is a Noetherian domain, $\mathrm{Int}(A)$ is a Prüfer domain if and only if A is a Dedekind domain with finite residue fields [5, Corollary 6.5].

In the general case, we have shown that if $\mathrm{Int}(A)$ is a Prüfer domain, then A is an almost Dedekind domain with finite residue fields [5, Proposition 6.3]. Recall that A is an *almost Dedekind domain* if $A_{\mathfrak{m}}$ is a rank-one discrete valuation domain for each maximal ideal $\mathfrak{m}$ of A [7]. The problem of determining conditions under which $\mathrm{Int}(A)$ is Prüfer has not been resolved.

In [9] Gilmer shows that various classical examples of non-Noetherian almost Dedekind domains do not have the finite residue fields property, and hence $\mathrm{Int}(A)$ is not Prüfer (as indeed he even proves that $\mathrm{Int}(A) = A[X]$). However, using a theorem of Krull [11, Theorem 3] concerning extensions of valuations,

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he gives a construction of non-Noetherian almost Dedekind domains with finite residue fields, yielding both examples where $\mathrm{Int}(A)$ is Prüfer and where it is not. Then he states two open questions, the second being closely related to his construction.

Q_4. If A is an almost Dedekind domain such that $\{|A/\mathfrak{m}| \mid \mathfrak{m} \in \mathrm{Max}(A)\}$ is bounded, is $\mathrm{Int}(A)$ a Prüfer domain?

Q_5. Suppose A_0 is a semilocal principal ideal domain with quotient field K_0 and K is an infinite algebraic extension of K_0 that is expressed as the union of a strictly ascending sequence $\{K_i\}$ of finite algebraic extensions of K_0. Let A_i be the integral closure of A_0 in K_i and A the union of the A_i. If $\mathrm{Int}(A)$ is a Prüfer domain, must there exist $N \in \mathbf{N}$ such that, for all i, j with $N \leq i \leq j$, $\mathrm{Int}(A_i) \subset \mathrm{Int}(A_j)$?

Throughout this paper we will generally assume that A is an almost Dedekind domain with finite residue fields and with quotient field K; we will try to find necessary or sufficient conditions for $\mathrm{Int}(A)$ to be Prüfer.

In the first section we determine necessary conditions on every subfield K_0 of K such that K/K_0 is a countably generated algebraic extension. This allows us to answer Q_4 negatively, but raises a new question:

Q_6. Are these conditions sufficient?

In a second section we determine a sufficient condition: if A is an almost Dedekind domain with finite residue fields and if, for each maximal ideal $\mathfrak{m}$ of A, $\mathrm{Int}(A)_\mathfrak{m} = \mathrm{Int}(A_\mathfrak{m})$ (we will say that $\mathrm{Int}(A)$ behaves well under localization), then $\mathrm{Int}(A)$ is a Prüfer domain. This raises another question:

Q_7. Is it necessary for $\mathrm{Int}(A)$ to behave well under localization to be Prüfer?

Next we state a necessary condition for good behaviour of localization called the *immediate subextension property.*

In the fourth section we restrict ourselves to the case where there is a subfield K_0 of K such that K/K_0 is a countably generated algebraic extension and the intersection $A_0 = A \cap K_0$ is a Dedekind domain with finite residue fields and quotient field K_0. We show that in this case the immediate subextension property is equivalent to good behaviour of localization.

In the following section we assume, moreover, that A is the integral closure of A_0 in K. We then show that Gilmer's condition in question Q_5 is necessary and sufficient for $\mathrm{Int}(A)$ to behave well under localization. We show also that if K is a normal extension of K_0, then $\mathrm{Int}(A)$ is Prüfer if and only if A is an almost Dedekind domain with finite residue fields.

In the sixth and last section we give examples using Gilmer's construction; the first one provides a negative answer to Q_4, another shows that, when A_0 is not semilocal, the answer to Q_5 may be negative, and yet another shows that the answers to Q_6 and Q_7 are not both affirmative.

1. Necessary conditions

First, recall that

1.1 [5, Proposition 6.3]. *If* $\mathrm{Int}(A)$ *is a Prüfer domain, then* A *is an almost Dedekind domain with finite residue fields.*

If $\mathfrak{m}$ is a maximal ideal of an almost Dedekind domain A, then $A_\mathfrak{m}$ is the ring of a rank-one discrete valuation $v_\mathfrak{m}$. If K_0 is a subfield of K such that K/K_0 is a countably generated algebraic extension, then the restriction of $v_\mathfrak{m}$ to K_0 is a rank-one discrete valuation and the valuation ring of this restriction $v_\mathfrak{m}|K_0$ is $A_\mathfrak{m} \cap K_0$. Let $e_\mathfrak{m}(K/K_0)$ be the ramification index of $v_\mathfrak{m}$ over $v_\mathfrak{m}|K_0$.

Similar to Gilmer's residue field condition in question Q_4:

(α) $\{|A/\mathfrak{m}|\,|\mathfrak{m} \in \mathrm{Max}(A)\}$ is bounded,

we propose the following ramification condition:

(β) For any subfield K_0 of K such that K/K_0 is a countably generated algebraic extension, $\{e_\mathfrak{m}(K/K_0)|\mathfrak{m} \in \mathrm{Max}(A)\}$ is bounded.

Neither condition is necessary in such a global version: for the first one consider $A = \mathbf{Z}$ and observe that $\mathrm{Int}(\mathbf{Z})$ is Prüfer; for the second one see Example 6.6. However, the next theorem shows that both weaker local versions are necessary; but each of them separately is not sufficient, and this will allow us to answer Gilmer's question Q_4 negatively.

1.2. **Theorem.** *Suppose* $\mathrm{Int}(A)$ *is a Prüfer domain. Let* K_0 *be a subfield of* K *such that* K/K_0 *is a countably generated algebraic extension. Then for each maximal ideal* $\mathfrak{m}$ *of* A:

(a) $\{|A/\mathfrak{n}|\ \ |\mathfrak{n} \in \mathrm{Max}(A),\ v_\mathfrak{n}|K_0 = v_\mathfrak{m}|K_0\}$ *is bounded, and*

(b) $\{e_\mathfrak{n}(K/K_0)|\mathfrak{n} \in \mathrm{Max}(A),\ v_\mathfrak{n}|K_0 = v_\mathfrak{m}|K_0\}$ *is bounded.*

Condition (a) is also given by Gilmer [9, Theorem 13]. First let us recall two results:

1.3 [3, Corollary, p. 303]. *For each maximal ideal* $\mathfrak{m}$ *of* A, $\mathrm{Int}(A)_\mathfrak{m}$ *is contained in the ring* $\mathrm{Int}(A_\mathfrak{m})$ *of integer-valued polynomials on* $A_\mathfrak{m}$.

1.4 ([2, Proposition 2] or [4, Lemma 1]). *Let* V *be the ring of a rank-one discrete valuation* v *with finite residue field of cardinal* q, *and let* P *be an integer-valued polynomial on* V *of degree* $d > 0$. *Then* $v(P) > -d/(q-1)$, *where* $v(P)$ *denotes the infimum of the values of the coefficients of* P.

In fact, Lemma 1 of [4] shows that the A-module $\mathrm{Int}(A)$ is generated by polynomials $Q_d(X)$, where $\deg(Q_d) = d$ and $v(Q_d) = \sum_{s>0}[d/q^s]$. Assertion 1.4 follows from the inequality $\sum_{s>0}[d/q^s] < \sum_{s>0}(d/q^s) = d/(q-1)$, for each $d > 0$.

1.5. **Lemma.** *Let* A *be an almost Dedekind domain and let* K_0 *be a subfield of* K *such that* K/K_0 *is an algebraic extension. Suppose there exists a rank-one discrete valuation* v_0 *of* K_0 *such that one of the following conditions fails*:

(a) $\{|A/\mathfrak{m}|\ \ |\mathfrak{m} \in \mathrm{Max}(A),\ v_\mathfrak{m}|K_0 = v_0\}$ *is bounded.*

(b) $\{e_\mathfrak{m}(K/K_0)|\mathfrak{m} \in \mathrm{Max}(A),\ v_\mathfrak{m}|K_0 = v_0\}$ *is bounded.*

Then $\mathrm{Int}(A) \cap K_0[X]$ *is contained in* $V_0[X]$, *where* V_0 *is the valuation ring of* v_0.

Proof. Let Q be a nonconstant polynomial belonging to $\mathrm{Int}(A) \cap K_0[X]$, let d be its degree, and choose a maximal ideal $\mathfrak{m}$ of A such that $v_\mathfrak{m}|K_0 = v_0$ and either $|A/\mathfrak{m}| > d$ or $e_\mathfrak{m}(K/K_0) > d$ (according to the failing condition). Since $Q(A) \subset A$, then $Q(A_\mathfrak{m}) \subset A_\mathfrak{m}$ (1.3); we will show that $v_\mathfrak{m}(Q) \geq 0$.

If $|A/\mathfrak{m}|$ is infinite, then clearly Q is in $A_\mathfrak{m}[X]$ [3, Proposition 5, Corollary 2]; if not we let $q = |A/\mathfrak{m}|$.

If $|A/\mathfrak{m}| > d$, then $v_\mathfrak{m}(Q) > -(d/q - 1) \geq -1$ (1.4), hence $v_\mathfrak{m}(Q) \geq 0$.

If $e_\mathfrak{m}(K/K_0) > d$, then $v_\mathfrak{m}(Q) > -(d/q - 1) \geq -d$, but Q belongs to $K_0[X]$ and $v_\mathfrak{m}(Q)$ is a multiple of $e_\mathfrak{m}(K/K_0)$, hence again $v_\mathfrak{m}(Q) \geq 0$. In any case, Q belongs to $A_\mathfrak{m}[X] \cap K_0[X] = V_0[X]$.

1.6. **Lemma.** *Let A be an almost Dedekind domain and let K_0 be a subfield of K such that K/K_0 is a countably generated algebraic extension. Suppose there exists a maximal ideal $\mathfrak{m}$ of A such that one of the following conditions fails*:

(a) $\{|A/\mathfrak{n}| \;\; |\mathfrak{n} \in \mathrm{Max}(A),\ v_\mathfrak{n}|K_0 = v_\mathfrak{m}|K_0\}$ *is bounded.*

(b) $\{e_\mathfrak{n}(K/K_0)|\mathfrak{n} \in \mathrm{Max}(A),\ v_\mathfrak{n}|K_0 = v_\mathfrak{m}|K_0\}$ *is bounded.*

Then there exists a maximal ideal $\mathfrak{n}$ of A such that $\mathrm{Int}(A)$ *is contained in* $A_\mathfrak{n}[X]$.

Proof. Let $\{K_j\}$ be an ascending sequence of finite extensions of K_0 such that $\bigcup_j K_j = K$. Let $\mathfrak{n}_0 = \mathfrak{m}$ and let v_0 be the restriction to K_0 of the valuation associated to $\mathfrak{m}$. For each $j > 0$ we define a maximal ideal $\mathfrak{n}_j$ of A in the following way: (i) one of the conditions (a) or (b) fails with respect to the extension K/K_j and the ideal $\mathfrak{n}_j$; (ii) $v_j|K_{j-1} = v_{j-1}$ where v_j denotes the restriction to K_j of the valuation associated to $\mathfrak{n}_j$. We can define such a sequence of maximal ideals $\mathfrak{n}_j$ since, for each $j > 0$, there are only finitely many valuations of K_j extending v_{j-1}.

Let v be the rank-one valuation of K whose restriction to each K_j is v_j. Let V be the valuation ring of v and V_j the valuation ring of v_j. By construction $A \cap K_j$ is contained in V_j and $A = \bigcup_j (A \cap K_j)$ is contained in $\bigcup_j V_j = V$. Hence there is a maximal ideal $\mathfrak{n}$ of A such that $A_\mathfrak{n} = V$ since A is an almost Dedekind domain. Now let Q be any element of $\mathrm{Int}(A)$ and let j be an integer such that Q belongs to $K_j[X]$. Lemma 1.5 implies that Q belongs to $V_j[X]$ and hence Q belongs to $A_\mathfrak{n}[X]$.

Proof of Theorem 1.2. If $\mathrm{Int}(A)$ is Prüfer, then A is an almost Dedekind domain (1.1). Suppose there exists a subfield K_0 of K such that K/K_0 is a countably generated algebraic extension and there exists a maximal ideal $\mathfrak{m}$ of A such that one of conditions (a) or (b) fails. Then $\mathrm{Int}(A)$ is contained in $A_\mathfrak{n}[X]$ for some maximal ideal $\mathfrak{n}$ of A (Lemma 1.6). Every overring of the Prüfer domain $\mathrm{Int}(A)$ is Prüfer; but $A_\mathfrak{n}[X]$ is not Prüfer since $A_\mathfrak{n}$ is not a field. This is a contradiction.

Note that if conditions (a) and (b) of Theorem 1.2 are satisfied for a subfield K_0 of K, then they are also satisfied for any subfield K_1 of K containing K_0. In the last paragraph we give examples of almost Dedekind domains A, which are integral extensions of the discrete valuation domain $\mathbf{Z}_p$ (with $p = 2$); hence $K_0 = \mathbf{Q}$ is the smallest subfield of K. In Example 6.2, condition (a) is satisfied (and even condition (α) of Gilmer), but not (b); in Example 6.3 condition (b) is satisfied (and even condition (β)) but not (a). Therefore the domains $\mathrm{Int}(A)$ are not Prüfer. Thus neither condition (α) nor (β) separately is sufficient, and in particular,

1.7. **Corollary.** *The answer to question* Q_4 *is negative.*

1.8. **Question** Q_6. Let A be an almost Dedekind domain with quotient field K and K_0 a subfield of K such that K/K_0 is an infinite countably generated algebraic extension. If for each maximal ideal $\mathfrak{m}$ of A:

(a) $\{|A/\mathfrak{n}| \ |\mathfrak{n} \in \mathrm{Max}(A),\ v_{\mathfrak{n}}|K_0 = v_{\mathfrak{m}}|K_0\}$ is bounded, and

(b) $\{e_{\mathfrak{n}}(K/K_0)|\mathfrak{n} \in \mathrm{Max}(A),\ v_{\mathfrak{n}}|K_0 = v_{\mathfrak{m}}|K_0\}$ is bounded,

is $\mathrm{Int}(A)$ a Prüfer domain?

2. Localization

Now, in order to get sufficient conditions, we consider localization properties.

2.1. **Theorem.** *Let A be an almost Dedekind domain with finite residue fields. If, for each maximal ideal $\mathfrak{m}$ of A, $\mathrm{Int}(A)_{\mathfrak{m}} = \mathrm{Int}(A_{\mathfrak{m}})$, then $\mathrm{Int}(A)$ is a Prüfer domain.*

Proof. We have to show that, for each maximal ideal $\mathfrak{P}$ of $\mathrm{Int}(A)$, $\mathrm{Int}(A)_{\mathfrak{P}}$ is a valuation domain. If $\mathfrak{P} \cap A$ is a maximal ideal $\mathfrak{m}$ of A, then $\mathrm{Int}(A)_{\mathfrak{P}} = (\mathrm{Int}(A)_{\mathfrak{m}})_{\mathfrak{P}_{\mathfrak{m}}}$ and it is enough to show that $\mathrm{Int}(A)_{\mathfrak{m}}$ is a Prüfer domain. If $\mathfrak{P} \cap A = (0)$, then, for any maximal ideal $\mathfrak{m}$ of A, $\mathrm{Int}(A)_{\mathfrak{P}} = (\mathrm{Int}(A)_{\mathfrak{m}})_{\mathfrak{P}_{\mathfrak{m}}}$ and it is also enough to show that $\mathrm{Int}(A)_{\mathfrak{m}}$ is a Prüfer domain. This results from 2.2 below, since, for each maximal ideal $\mathfrak{m}$ of A, $A_{\mathfrak{m}}$ is a rank-one discrete valuation domain with finite residue field and $\mathrm{Int}(A)_{\mathfrak{m}} = \mathrm{Int}(A_{\mathfrak{m}})$ is Prüfer.

2.2 [4, Propositions 1 and 2]. *If V is a rank-one discrete valuation domain with finite residue field and with quotient field K, then:*

(i) *the prime ideals of* $\mathrm{Int}(V)$ *lying over the ideal* (0) *of* V *are the ideals* $\mathfrak{P}_S = \{Q \in \mathrm{Int}(V) | Q = S \cdot R,\ R \in K[X]\}$ *where S is an irreducible polynomial of* $K[X]$;

(ii) *the prime ideals of* $\mathrm{Int}(V)$ *lying over the maximal ideal* $\mathfrak{m}$ *of* V *are the maximal ideals* $\mathfrak{m}_\alpha(V) = \{Q \in \mathrm{Int}(V) | Q(\alpha) \in \mathfrak{m}V^*\}$, *where V^* is the completion of V and α is any element of V^**;

(iii) *the localization of* $\mathrm{Int}(V)$ *with respect to* $\mathfrak{P}_S$ *is the valuation ring* $K[X]_{(S)}$ *and the localization of* $\mathrm{Int}(V)$ *with respect to* $\mathfrak{m}_\alpha(V)$ *is the valuation ring* $V_\alpha = \{R \in K(X) | R(\alpha) \in V^*\}$.

We will say that $\mathrm{Int}(A)$ behaves well under localization if the condition of Theorem 2.1 is fulfilled. Note that it is always the case when A is Noetherian [3, p. 303]; this yields the well-known fact:

2.3 [5, Corollary 6.5]. *If A is Noetherian,* $\mathrm{Int}(A)$ *is Prüfer if and only if A is a Dedekind domain with finite residue fields.*

We know that $\mathrm{Int}(A)$ does not always behave well under localization when A is an almost Dedekind domain with finite residue fields [9, Theorem 13]. Conversely we ask:

2.4. **Question** Q_7. For $\mathrm{Int}(A)$ to be Prüfer, is it necessary that it behaves well under localization?

2.5. *Remark.* If $\mathrm{Int}(A)$ is a Prüfer domain, then, for each maximal ideal $\mathfrak{m}$ of A and each element α of the completion $A^*_{\mathfrak{m}}$ of $A_{\mathfrak{m}}$, the localization of $\mathrm{Int}(A)$ with respect to $\mathfrak{m}_\alpha = \{Q \in \mathrm{Int}(A) | Q(\alpha) \in \mathfrak{m}A^*_{\mathfrak{m}}\}$ is the valuation domain $V_\alpha = \{R \in K(X) | R(\alpha) \in A^*_{\mathfrak{m}}\}$. To see this, note that the domain $\mathrm{Int}(A)$ is contained in $\mathrm{Int}(A_{\mathfrak{m}})$ (1.3) and the ideal $\mathfrak{m}_\alpha$ is the intersection of $\mathrm{Int}(A)$ with the ideal $(\mathfrak{m}A_{\mathfrak{m}})_\alpha$ of $\mathrm{Int}(A_{\mathfrak{m}})$. If $\mathrm{Int}(A)$ is a Prüfer domain, then the localization of $\mathrm{Int}(A)$ with respect to $\mathfrak{m}_\alpha$ is a valuation domain. This latter domain is contained in the valuation domain V_α and its maximal ideal is contained in the maximal ideal of V_α. Thus these valuation domains coincide. So when $\mathrm{Int}(A)$ is Prüfer, every localization of $\mathrm{Int}(A_\mathfrak{m})$ is a localization of $\mathrm{Int}(A)$. Thus an equivalent form of Q_7 asks whether every prime ideal of $\mathrm{Int}(A)$ lying over a maximal ideal $\mathfrak{m}$ of A is an ideal $\mathfrak{m}_\alpha$.

3. Immediate subextension property

Now we state a necessary condition for $\mathrm{Int}(A)$ to behave well under localization. Recall that the *prime field* of a field K is the smallest subfield contained in K; it is isomorphic to $\mathbf{Q}$ or to $\mathbf{F}_p$.

3.1. **Proposition.** *Let $\mathfrak{m}$ be a maximal ideal of A such that $A_\mathfrak{m}$ is a rank-one discrete valuation domain with finite residue field. If $\mathrm{Int}(A)_\mathfrak{m} = \mathrm{Int}(A_\mathfrak{m})$, then there exists a subfield K_1 of K, finitely generated over the prime field of K, such that, if $A_1 = A \cap K_1$ and $\mathfrak{m}_1 = \mathfrak{m} \cap K_1$, then for each maximal ideal $\mathfrak{n}$ of A lying over $\mathfrak{m}_1$,*

$$\mathfrak{n}A_\mathfrak{n} = \mathfrak{m}_1 A_\mathfrak{n} \quad \textit{and} \quad A/\mathfrak{n} \cong A_1/\mathfrak{m}_1 .$$

Proof. We first construct the field K_1. Let $a_0, \ldots, a_{q-1}$ be a complete set of residues of $\mathfrak{m}$ in A and let t be a local parameter of v, which belongs to A. The polynomial $P = (X-a_0)\cdots(X-a_{q-1})/t$ belongs to $\mathrm{Int}(A_\mathfrak{m})$ and also to $\mathrm{Int}(A)_\mathfrak{m}$ by hypothesis. Let s be an element of $A\backslash\mathfrak{m}$ such that sP belongs to $\mathrm{Int}(A)$ and define K_1 to be the subfield of K generated over $\mathbf{Q}$ or $\mathbf{F}_p$ by $a_0, \ldots, a_{q-1}$, t, and s.

If we set $A_1 = A \cap K_1$ and $\mathfrak{m}_1 = \mathfrak{m} \cap K_1$, then $A_1/\mathfrak{m}_1 \cong A/\mathfrak{m}$ and $tA_1 \subset \mathfrak{m}_1$. Let $\mathfrak{n}$ be a maximal ideal of A such that $\mathfrak{n} \cap A_1 = \mathfrak{m} \cap A_1 = \mathfrak{m}_1$. Let a be any element of A. Since $sP(a)$ belongs to A, $stP(a) = s(a-a_0)\cdots(a-a_{q-1})$ belongs to $tA \subset \mathfrak{n}$. The element s of A_1 does not belong to $\mathfrak{m}$, hence it does not belong to $\mathfrak{n}$, and there exists i such that $a - a_i$ belongs to $\mathfrak{n}$. Therefore $A/\mathfrak{n} \cong A_1/\mathfrak{m}_1$. Let b be any element of $\mathfrak{n}$. Since $sP(b+a_0)$ is in A, $stP(b+a_0) = sb(b+a_0-a_1)\cdots(b+a_0-a_{q-1})$ is in tA; but $s(b+a_0-a_1)\cdots(b+a_0-a_{q-1})$ is in the multiplicative system $A\backslash\mathfrak{n}$, hence b belongs to $tA_\mathfrak{n}$. Therefore $\mathfrak{m}_1 A_\mathfrak{n}$ contains $\mathfrak{n}$ and $\mathfrak{m}_1 A_\mathfrak{n} \cong \mathfrak{n}A_\mathfrak{n}$.

Recall that a *valuation* v of K is *essential* for the domain A if the valuation ring of v is the localization of A with respect to a prime ideal $\mathfrak{m}$. Recall also that an extension v of a valuation v_1 of a field K_1 is an *immediate extension* if v and v_1 have the same value group and same residue field; in this case we will say that v is *immediate over* K_1.

If A_n is the ring of a valuation v, letting v_1 be the restriction of v to K_1, the conditions $\mathfrak{n}A_\mathfrak{n} = \mathfrak{m}_1 A_\mathfrak{n}$ and $A/\mathfrak{n} \cong A_1/\mathfrak{m}_1$ of Proposition 3.1 imply that v is immediate over K_1. We then make the following definitions:

3.2. **Definitions.** Let A be a Prüfer domain with quotient field K.

(i) A valuation v of K, which is essential for A, is said to be *totally A-immediate over* a subextension K_1 if, letting v_1 be the restriction of v to K_1, each extension w of v_1 to K, which is essential for A, is immediate over K_1.

(ii) The domain A is said to have the *immediate subextension property* over a subfield K_0 of K if, for each valuation v, which is essential for A, there exists a subextension K_1 of K finitely generated over K_0, over which v is totally A-immediate.

If the valuation v of K is totally A-immediate over a subfield K_0, then v is totally A-immediate over every subfield K_1 containing K_0, and, if A has the immediate subextension property over a subfield K_0 of K, then A has the immediate subextension property over every subfield K_1 of K containing K_0.

3.3. **Theorem.** *Suppose A is an almost Dedekind domain with finite residue fields. If, for each maximal ideal $\mathfrak{m}$ of A, $\mathrm{Int}(A)_{\mathfrak{m}} = \mathrm{Int}(A_{\mathfrak{m}})$, then A has the immediate subextension property over the prime field of K.*

The theorem results from Proposition 3.1.

4. Partial converse

We now restrict ourselves to the case where there is a subfield K_0 of K such that K/K_0 is a countably generated algebraic extension and the intersection $A_0 = A \cap K_0$ is a Dedekind domain with finite residue fields and quotient field K_0. We will show that the immediate subextension property is then equivalent to good behaviour of localization; we start with a lemma under somewhat more general conditions.

4.1. **Lemma.** *Let A be a Prüfer domain with quotient field K. Suppose that K_0 is a subfield of K such that K/K_0 is a countably generated algebraic extension. The following assertions are equivalent:*

(i) *A has the immediate subextension property over K_0.*

(ii) *For each valuation v_0 of K_0, which is the restriction of a valuation essential for A, there exists a finite extension K' of K_0 contained in K such that each extension of v_0 to K, which is essential for A, is immediate over K'.*

Proof. It is clear that (ii) implies (i). Now suppose (ii) does not hold and write K as the union of an ascending sequence of finite extensions K_n of K_0. Then (ii) does not hold for at least one extension v_1 of v_0 to K_1, which is the restriction of an essential valuation for A; to see this, assume to the contrary that, for each extension w of v_0 to K_1, which is the restriction of an essential valuation for A, there is an integer $j(w)$ such that each extension of w to K, essential for A, is immediate over $K_{j(w)}$; then taking j to be the supremum of these integers $j(w)$ (since there are only finitely many extensions of v_0 to K_1), each extension of v_0 to K, essential for A, would be immediate over K_j. By induction it is then possible to construct a sequence (v_n) of valuations, v_n extending v_{n-1} to K_n, such that there exists an extension w_n of v_n to K, essential for A, which is not an immediate extension of v_n. This sequence defines a valuation v of K, extending v_0, and v is essential for A. Indeed if a is an element of A, then it belongs to a field K_n, and by construction $v(a) = v_n(a) = w_n(a) \geq 0$. Hence the ring of v contains A. Now for each n, v is not totally A-immediate over K_n, hence it is not totally A-immediate over any finite extension of K_0.

4.2. *Remark.* If A is an almost Dedekind domain, it follows from Lemma 4.1 that the immediate subextension property over a subfield K_0 of K such that K/K_0 is a countably generated algebraic extension is stronger than conditions (a) and (b) of Theorem 1.2.

We are now ready for the main theorem of this section.

4.3. **Theorem.** *Let A be an almost Dedekind domain with finite residue fields. If the quotient field K of A is a countably generated algebraic extension of a field K_0 and if the ring $A_0 = A \cap K_0$ is a Dedekind domain with finite residue fields and with quotient field K_0, then the following conditions are equivalent*:

(i) *For each maximal ideal $\mathfrak{m}$ of A, $\mathrm{Int}(A)_{\mathfrak{m}} = \mathrm{Int}(A_{\mathfrak{m}})$.*

(ii) *A has the immediate subextension property over K_0.*

(iii) *A has the immediate subextension property over the prime field of K.*

Proof. Theorem 3.3 shows that (i) implies (iii), and it is trivial that (iii) implies (ii). Conversely, if (ii) holds, let $\mathfrak{m}$ be a maximal ideal of A and P a polynomial in $\mathrm{Int}(A_{\mathfrak{m}})$. We have to show that P belongs to $(\mathrm{Int}(A))_{\mathfrak{m}}$, i.e., that $P = Q/s$, where s belongs to $A \backslash \mathfrak{m}$ and Q to $\mathrm{Int}(A)$.

Let K^0 be a finite extension of K_0 such that P belongs to $K^0[X]$. Let A^0 be the domain $A \cap K^0$; A^0 is a Dedekind domain since it contains the integral closure of A_0 in K^0, which is a Dedekind domain. Let d be a nonzero element of A^0 such that the coefficients of dP are in A^0. Then d belongs to only a finite number of maximal ideals $\mathfrak{p}_k$ of A^0. If no $\mathfrak{p}_k$ is contracted from a maximal ideal of A, then d is a unit of A and the desired conclusion holds. For each $\mathfrak{p}_k$ lying under some maximal ideal of A, let $K(\mathfrak{p}_k)$ be a finite extension of K^0 such that each extension of the valuation associated to $\mathfrak{p}_k$ in K^0 to a valuation on K that is essential for A is immediate over $K(\mathfrak{p}_k)$ (Lemma 4.1). Define K^* to be the finite extension of K^0 generated by these fields $K(\mathfrak{p}_k)$.

Let A^* be the Dedekind domain $A \cap K^*$ and let $\mathfrak{m}^*$ be the maximal ideal $\mathfrak{m} \cap K^*$ of A^*. The localization of A^* with respect to $\mathfrak{m}^*$ is the intersection of $A_{\mathfrak{m}}$ with K^*. By hypothesis $P(A_{\mathfrak{m}})$ is contained in $A_{\mathfrak{m}}$ so that $P(A^*_{\mathfrak{m}^*})$ is contained in $A_{\mathfrak{m}} \cap K^* = (A^*)_{\mathfrak{m}^*}$ and P belongs to $\mathrm{Int}(A^*_{\mathfrak{m}^*})$. But $\mathrm{Int}(A^*_{\mathfrak{m}^*}) = \mathrm{Int}(A^*)_{\mathfrak{m}^*}$, because A^* is Noetherian [3, Corollary 5, p. 303]; hence $P = Q/s$, where s belongs to $A^* \backslash m^*$ and the polynomial Q is such that $Q(A^*) \subset A^*$.

It remains to prove that $Q(A) \subset A$, or equivalently, that for each maximal ideal $\mathfrak{n}$ of A, $Q(A) \subset A_{\mathfrak{n}}$:

If $d \notin \mathfrak{n}$, then $dQ = dsP$ has its coefficients in A, and clearly $Q(A) \subset A_{\mathfrak{n}}$.

If $d \in \mathfrak{n}$, then let $\mathfrak{n}^* = \mathfrak{n} \cap A^*$. By construction $\mathfrak{n}A_{\mathfrak{n}} = \mathfrak{n}^* A_{\mathfrak{n}}$ and $A/\mathfrak{n} \cong A^*/\mathfrak{n}^*$. From $Q(A^*) \subset A^*$ it follows that $Q(A^*_{\mathfrak{n}^*}) \subset A^*_{\mathfrak{n}^*}$ [3], and this implies that $Q(A_{\mathfrak{n}}) \subset A_{\mathfrak{n}}$ (Proposition 4.4 below).

4.4. **Proposition** [6, Proposition 5.5]. *Let R be a Noetherian local domain with maximal ideal $\mathfrak{m}$ and quotient field K. Let R_0 be a local subring of R with maximal ideal $\mathfrak{m}_0$. If $\mathfrak{m}_0 R = \mathfrak{m}$ and $R/\mathfrak{m} \cong R_0/\mathfrak{m}_0$, then*

$$\mathrm{Int}(R) = \{P \in K[X] \| P(R_0) \subset R\}.$$

Note that, in order to prove Theorem 4.3, we only use the previous result in the case where R_0 is a discrete valuation domain with finite residue field. In this case, R is an immediate extension of R_0 and there is a basis of the R-module $\mathrm{Int}(R)$ whose elements are polynomials belonging to $\mathrm{Int}(R_0)$ [4, Lemma 1], and hence $\mathrm{Int}(R_0) \subset \mathrm{Int}(R)$.

Thus, with the hypothesis of Theorem 4.3, we have proved the following implications:

$$
\begin{tikzcd}[column sep=large]
\boxed{\begin{array}{c}\mathrm{Int}(A)\text{ behaves well}\\ \text{under localization}\end{array}} \arrow[r, shift left, "\text{Theorem 3.3}"] \arrow[d, shift right, "\text{Theorem 2.1}"'] & \boxed{\begin{array}{c}A\text{ has the immediate}\\ \text{subextension property}\end{array}} \arrow[l, shift left, "\text{Theorem 4.3}"] \arrow[d, shift right] \\
\boxed{\mathrm{Int}(A)\text{ is Prüfer}} \arrow[u, dashed, shift right, "Q_7?"'] \arrow[r, shift left, "\text{Theorem 1.2}"] & \boxed{\text{conditions (a) and (b)}} \arrow[l, dashed, shift left, "Q_6?"] \arrow[u, shift right, "/" marking, "\text{Remark 4.2}"']
\end{tikzcd}
$$

4.5. **Corollary.** *Let A be an integrally closed domain such that the quotient field K of A is a countably generated algebraic extension of a field K_0 and the ring $A_0 = A \cap K_0$ is a Dedekind domain with finite residue fields and with quotient field K_0. If A has the immediate subextension property over K_0, then*:

(i) *A is an almost Dedekind domain with finite residue fields.*
(ii) *For each maximal ideal $\mathfrak{m}$ of A,* $\mathrm{Int}(A)_\mathfrak{m} = \mathrm{Int}(A_\mathfrak{m})$.
(iii) $\mathrm{Int}(A)$ *is a Prüfer domain.*

Proof. In view of Theorem 4.3 and the diagram above, it suffices to prove (i). Note that A is a Prüfer domain since A is an overring of the integral closure of A_0 in K, which is a Prüfer domain. The immediate subextension property then implies that A is an almost Dedekind domain with finite residue fields.

5. Integral closure of a Dedekind domain

Let A_0 be a Dedekind domain with finite residue fields, K_0 the quotient field of A_0, K a countably generated algebraic extension of K_0, and A the integral closure of A_0 in K. Under these hypotheses A is a Prüfer domain, and if A has the immediate subextension property, it is an almost Dedekind domain with finite residue fields.

5.1. **Proposition.** *Let A_0 be a Dedekind domain with quotient field K_0, K a countably generated algebraic extension of K_0, and A the integral closure of A_0 in K. Suppose that A is an almost Dedekind domain with finite residue fields. The following conditions are equivalent*:

(i) *For each maximal ideal $\mathfrak{m}$ of A,* $\mathrm{Int}(A)_\mathfrak{m} = \mathrm{Int}(A_\mathfrak{m})$.
(ii) *A has the immediate subextension property over K_0.*
(iii) *For each valuation v_0 of K_0, which is essential for A_0, there exists a finite extension K_1 of K_0 such that each extension v of v_0 to K, which is essential for A, is immediate over K_1.*

Proof. The equivalence (i) $\leftrightarrow$ (ii) results from Theorem 4.3. Lemma 4.1 shows the equivalence (ii) $\leftrightarrow$ (iii) since each valuation v of K, which is essential for A, is the extension of—and has for restriction—a valuation v_0 of K_0, which is essential for A_0.

With regard to localization, the following corollary provides an affirmative answer to a question that is analogous to Gilmer's question Q_5 quoted in the introduction.

5.2. **Corollary.** *Let A_0 be a semilocal principal ideal domain with quotient field K_0. Let K be the union of an ascending sequence of finite algebraic extensions*

K_n of K_0. Let A be the integral closure of A_0 in K and, for each n, let $A_n = A \cap K_n$. If A is an almost Dedekind domain with finite residue fields, the following conditions are equivalent:

(i) *For each maximal ideal $\mathfrak{m}$ of A, $\mathrm{Int}(A)_{\mathfrak{m}} = \mathrm{Int}(A_{\mathfrak{m}})$.*
(ii) *There exists n such that every valuation of K, which is essential for A, is immediate over K_n.*
(iii) *There exists n such that for all i, j with $n \le i \le j$, $\mathrm{Int}(A_i) \subset \mathrm{Int}(A_j) \subset \mathrm{Int}(A)$.*

Proof. (i) $\to$ (ii) results from Proposition 5.1 since A_0 is semilocal.

(ii) $\to$ (iii) Let n be an integer as in condition (ii) and let i and j be such that $n \le i \le j$. Let $\mathfrak{q}$ be a maximal ideal of A_j and let $\mathfrak{p} = \mathfrak{q} \cap A_i$. The valuation associated with the ring $(A_j)_{\mathfrak{q}}$ is an immediate extension of the valuation associated with the ring $(A_i)_{\mathfrak{p}}$; then

$$\mathrm{Int}(A_i) \subset \mathrm{Int}((A_i)_{\mathfrak{p}}) \subset \mathrm{Int}((A_j)_{\mathfrak{q}}) = \mathrm{Int}(A_j)_{\mathfrak{q}}$$

for each maximal ideal $\mathfrak{q}$ of A_j (Proposition 4.4). Therefore $\mathrm{Int}(A_i) \subset \mathrm{Int}(A_j)$.

Moreover, $\mathrm{Int}(A_i) \subset \mathrm{Int}(A)$; let Q be an element of $\mathrm{Int}(A_i)$. For each element a of A, there exists $j > i$ such that a belongs to A_j; thus $Q(a)$ belongs to A_j since $\mathrm{Int}(A_i) \subset \mathrm{Int}(A_j)$, and Q is in $\mathrm{Int}(A)$.

(iii) $\to$ (i) Let $\mathfrak{m}$ be a maximal ideal of A and let Q be any element of $\mathrm{Int}(A_{\mathfrak{m}})$. Let K_i be an extension of K_0 such that Q belongs to $K_i[X]$ and let $\mathfrak{p}$ be the ideal $\mathfrak{m} \cap A_i$. By hypothesis $Q(A_{\mathfrak{m}})$ is contained in $A_{\mathfrak{m}}$, so that $Q((A_i)_{\mathfrak{p}}) \subset A_{\mathfrak{m}} \cap K_i = (A_i)_{\mathfrak{p}}$ and Q belongs to $\mathrm{Int}((A_i)_{\mathfrak{p}}) = \mathrm{Int}(A_i)_{\mathfrak{p}}$. Hence $Q = P/s$, where s is an element of $A_i \backslash \mathfrak{p}$ and P belongs to $\mathrm{Int}(A_i)$. Since $\mathrm{Int}(A_i)$ is contained in $\mathrm{Int}(A)$, Q belongs to $\mathrm{Int}(A)_{\mathfrak{m}}$.

If K is a normal algebraic extension of K_0, the four properties in the diagram at the end of §4 are satisfied as soon as A is an almost Dedekind domain with finite residue fields and we have a characterization of when $\mathrm{Int}(A)$ is a Prüfer domain:

5.3. **Theorem.** *Let A_0 be a Dedekind domain with finite residue fields and with quotient field K_0. Let K be a countably generated normal algebraic extension of K_0 and let A be the integral closure of A_0 in K. The following conditions are equivalent:*

(i) *A is an almost Dedekind domain with finite residue fields.*
(ii) *A has the immediate subextension property over K_0.*
(iii) *$\mathrm{Int}(A)$ is a Prüfer domain.*
(iv) *For each maximal ideal $\mathfrak{m}$ of A:*
  (a) *$\{|A/\mathfrak{n}| \mid \mathfrak{n} \in \mathrm{Max}(A),\ v_{\mathfrak{n}}|K_0 = v_{\mathfrak{m}}|K_0\}$ is bounded, and*
  (b) *$\{e_{\mathfrak{n}}(K/K_0) \mid \mathfrak{n} \in \mathrm{Max}(A),\ v_{\mathfrak{n}}|K_0 = v_{\mathfrak{m}}|K_0\}$ is bounded.*

If A_0 is semilocal, the equivalence of (i) and (iii) is also proved by Gilmer [9, Theorem 12].

Proof. Implications (ii) $\to$ (iii) $\to$ (iv) result from Theorems 2.1 and 1.2; (iv) $\to$ (i) is immediate. Let us prove (i) $\to$ (ii). Let v be a valuation of K, which is essential for A. Since v is discrete and has a finite residue field, there exists a finite extension K_1 of K_0 such that v is an immediate extension of its restriction v_1 to K_1 (K_1 is generated by a complete set of residues and by a local parameter of v). As the extension K/K_1 is normal, every extension of v_1 to K is also an immediate extension and v is totally A-immediate over K_1.

6. COUNTEREXAMPLES

In order to answer some questions (and in particular Gilmer's question Q$_4$) and shed some light on the others, we will give several examples. We construct rings of algebraic numbers, which are integral closures of $\mathbf{Z}$ or of localizations of $\mathbf{Z}$ in algebraic extensions of $\mathbf{Q}$. As Gilmer does, we use Hasse's existence theorem about prime ideal decomposition in algebraic number fields.

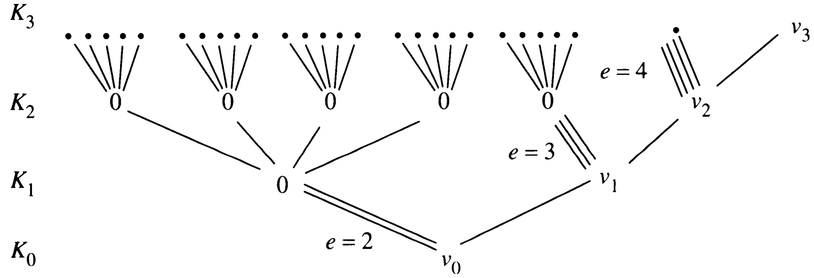
6.1 (Hasse [10]). Let K_0 be an algebraic number field and let $\mathfrak{m}_1, \dots, \mathfrak{m}_s$ be prime ideals of the ring A_0 of algebraic integers of K_0. Suppose given, for each $i = 1, \dots, s$, $2r(i)$ positive integers e_{ij} and f_{ij} $(j = 1, \dots, r(i))$ in such a way that $\sum_{1 \le j \le r(i)} e_{ij} f_{ij} = n$ (for every $i = 1, \dots, s$). Then there exists an algebraic extension K of K_0 having degree n such that each prime ideal $\mathfrak{m}_i$ decomposes in the field K as a product $\mathfrak{m}_i = \prod_{1 \le j \le r(i)} \mathfrak{M}_{ij}^{e_{ij}}$, where the $\mathfrak{M}_{ij}$ are prime ideals of the ring A of algebraic integers of K and $[A/\mathfrak{M}_{ij} : A_0/\mathfrak{m}_i] = f_{ij}$.

In every example that we are going to construct, K is the union of a strictly ascending sequence of finite extensions K_n of $K_0 = \mathbf{Q}$. We define every extension K_n/K_{n-1} with Hasse's existence theorem, where $e_{ij} = 1$ or $f_{ij} = 1$.

6.2. **Example.** Let v_0 be the 2-adic valuation of $K_0 = \mathbf{Q}$ and $A_0 = \mathbf{Z}_{(2)}$ the valuation ring of v_0. We define K_n by induction on n: K_n is an extension of K_{n-1} such that (i) $[K_n : K_{n-1}] = n + 2$; (ii) every valuation of K_{n-1}, which is ramified over the valuation v_0 of $\mathbf{Q}$ has only immediate extensions to K_n $(e = 1,\ f = 1)$; (iii) the valuation v_{n-1} of K_{n-1}, which is not ramified over v_0 has two extensions to K_n; one of these is totally ramified $(e = [K_n : K_{n-1}] - 1$, $f = 1)$, and the other, denoted by v_n, is immediate $(e = 1,\ f = 1)$.

Let K be the union of the extensions K_n and let A be the integral closure of $A_0 = \mathbf{Z}_{(2)}$ in K; A is also the intersection of the valuation rings of the extensions of v_0 to K.

A tree may represent the extensions of valuations as follows: a single line represents an immediate extension, a multiple line represents a ramified extension (as depicted in the figure).



A valuation of K is a branch of the tree; each extension of v_0 has relative degree one and a finite ramification index. So A is an almost Dedekind domain and each residue field of A is isomorphic to $\mathbf{F}_2$, while $\{v_{\mathfrak{m}}(2) | \mathfrak{m} \in \operatorname{Max}(A)\} = \mathbf{N}^*$. The condition (b) of Theorem 1.2 does not hold and we answer Gilmer's question Q_4 negatively:

> $\operatorname{Int}(A)$ is not a Prüfer domain and A is an almost Dedekind domain such that $\{|A/\mathfrak{m}| \ | \mathfrak{m} \in \operatorname{Max}(A)\} = \{2\}$ is bounded.

6.3. **Example** [9, Gilmer's Example 14]. We reverse the roles of e and f in Example 6.2: every valuation of K_{n-1} whose residue field is not isomorphic to $\mathbf{F}_2$ has only immediate extensions to K_n $(e = 1, f = 1)$ and the valuation v_{n-1} of K_{n-1} whose residue field is isomorphic to $\mathbf{F}_2$ has two extensions to K_n; one of these is such that $e = 1$ and $f = [K_n : K_{n-1}] - 1$, and the other, denoted by v_n, is immediate. Then A is an almost Dedekind domain with finite residue fields. For each maximal ideal $\mathfrak{m}$ of A, $v_{\mathfrak{m}}(2) = 1$, while $\{|A/\mathfrak{m}| \ | \mathfrak{m} \in \operatorname{Max}(A)\} = \{2^n | n \in \mathbf{N}^*\}$. Condition (a) of Theorem 1.2 is not fulfilled and $\operatorname{Int}(A)$ is not a Prüfer domain, but condition (b) is satisfied. Thus the condition that $\{v_{\mathfrak{m}}(\mathfrak{m} \cap A_0) | \mathfrak{m} \in \operatorname{Max}(A)\}$ is bounded is not sufficient for $\operatorname{Int}(A)$ to be Prüfer.

Now let us show that the answer to question Q_5 would be negative if we did not hypothesize A_0 to be a semilocal domain.

6.4. **Example.** Let $K_0 = \mathbf{Q}$ and $A_0 = \mathbf{Z}$. Letting p_n be the nth prime number, we define K_n by induction on n; K_n is an extension of K_{n-1} such that: (i) $[K_n : K_{n-1}] = 2$; (ii) every valuation of K_{n-1}, which is an extension of the 2-adic valuation or of the 3-adic valuation or ..., the p_{n-1}-adic valuation of $\mathbf{Q}$ is completely decomposed (has only immediate extensions to K_n); (iii) every valuation of K_{n-1}, which is an extension of the p_n-adic valuation of $\mathbf{Q}$, is totally ramified (has only one extension to K_n with $e = [K_n : K_{n-1}]$ and $f = 1$). [Note that for each step we only consider a finite number of valuations.]

Let K be the union of the K_n and A the integral closure of $\mathbf{Z}$ in K. The ring A has the immediate subextension property: for every n, each extension of the p_n-adic valuation is immediate over K_n. Hence, for each maximal ideal $\mathfrak{m}$ of A, $\operatorname{Int}(A)_{\mathfrak{m}} = \operatorname{Int}(A_{\mathfrak{m}})$ and $\operatorname{Int}(A)$ is Prüfer (Corollary 4.5). However, for $i < j$, each extension to K_j of the p_j-adic valuation is ramified over K_i, hence $\operatorname{Int}(A_i)$ is not included in $\operatorname{Int}(A_j)$ ([6, Proposition 5.3] or [9, Proposition 11]).

Let us show now that conditions (a) and (b) of Theorem 1.2 do not imply that $\operatorname{Int}(A)$ behaves well under localization.

6.5. **Example.** A slightly different version of Example 6.2: $[K_n : K_{n-1}] = 3$. Then A is an almost Dedekind domain with finite residue fields since every extension of the 2-adic valuation v_0 has relative degree one and ramification index two, except the valuation v, whose restriction to each K_n is v_n and which is an immediate extension of v_0. But, for each n, $v|K_n = v_n$ has ramified extensions to K, and the immediate extension v of v_0 contradicts the immediate subextension property.

Hence $\mathrm{Int}(A)$ does not behave well under localization although conditions (a) and (b) of Theorem 1.2 are fulfilled. We do not know if $\mathrm{Int}(A)$ is Prüfer, but if $\mathrm{Int}(A)$ is Prüfer the answer to Q_7 is negative and if $\mathrm{Int}(A)$ is not Prüfer the answer to Q_6 is negative.

We said that $\mathrm{Int}(\mathbf{Z})$ is Prüfer although $\{|\mathbf{Z}/\mathfrak{m}| \mid \mathfrak{m} \in \mathrm{Max}(\mathbf{Z})\}$ is not bounded; let us construct now an example A such that $\mathrm{Int}(A)$ is Prüfer and $\{v_{\mathfrak{m}}(\mathfrak{m} \cap \mathbf{Z}) \mid \mathfrak{m} \in \mathrm{Max}(A)\}$ is not bounded.

6.6. **Example.** A slightly different version of Example 6.4: $[K_n : K_{n-1}] = n+2$. The ring A has the immediate subextension property, hence $\mathrm{Int}(A)$ is Prüfer (Corollary 4.5) and conditions (a) and (b) of Theorem 1.2 are fulfilled, but condition (β) on ramification index is not.

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