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ON THE DIVERGENCE OF LAGRANGE INTERPOLATION WITH EQUIDISTANT NODES

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ABSTRACT. This paper is concerned with the optimal rate of divergence of Lagrange interpolation of $f(x) = |x|$ at equidistant nodes.

1. MAIN RESULTS

In this note we discuss a problem in the divergent aspect of Lagrange interpolation. Denote $x_{k,n} := -1+2(k-1)/(n-1)$, $k = 1, 2, \dots, n$, $n = 2, 3, \dots$. Recall that given a function $f(x)$ defined on $[-1, 1]$, the Lagrange interpolation polynomial $L_n(f; x)$, of degree at most $n-1$, is (uniquely) defined by the conditions

$$L_n(f; x_{k,n}) = f(x_{k,n}) \qquad (k = 1, 2, \dots, n; n = 2, 3, \dots).$$

The following divergent result of Bernstein is well known (cf. [N, p. 30]).

Theorem 1 (Bernstein, 1918). *For function $f(x) = |x|$, the sequence $\{L_n(f; x)$: $n = 2, 3, \dots\}$ diverges if $0 < |x| < 1$.*

Recently, Byrne, Mills, and Smith [BMS] considered the rate of this divergence process. More precisely, they proved

Theorem 2 (Byrne, Mills, and Smith, 1990 [BMS]). *For the function $f(x) = |x|$, we have*

$$\limsup_{n\to\infty} n^{-1} \log|L_n(f; x) - |x|| = \tfrac{1}{2}[(1+x)\log(1+x) + (1-x)\log(1-x)]$$

if $0 < |x| < 1$.

From now on, we will write $L_n(x) = L_n(f; x)$ if $f(x) = |x|$.

The result of Byrne, Mills, and Smith tells us that for every x with $0 < |x| < 1$, there exists a subsequence, say $\{L_n(x)$: $n = n_1, n_2, n_3, \dots\}$, of $\{L_n(x)$: $n = 2, 3, \dots\}$, whose rate of divergence is geometrically fast; but it seems that the sequence $\{n_1, n_2, n_3, \dots\}$ should depend on x. We will show that there actually exists a subsequence that works for almost all x (with $0 < |x| < 1$) as implied by the following results.

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Theorem 3. *For all $x \in R$, we have*

$$\lim_{n\to\infty}\left|\frac{L_n(x)-|x|}{W_n(x)}\right|^{1/n} = e,$$

where $W_n(x) := \prod_{k=1}^{n}(x - x_{k,n})$.

Corollary 4. *Let $\{p_k : k = 1, 2, 3, \dots\}$ be the sequence of all positive prime integers with $p_1 < p_2 < p_3 < \cdots$. Then for $0 < |x| < 1$, we have*

$$\begin{aligned}(1)\qquad \limsup_{n\to\infty} n^{-1}\log|L_n(x)-|x|| &= \limsup_{k\to\infty}(p_k+1)^{-1}\log|L_{p_k+1}(x)-|x||\\ &= \tfrac{1}{2}[(1+x)\log(1+x)+(1-x)\log(1-x)].\end{aligned}$$

Furthermore, if we denote $T := \{x \in [-1, 1] | \liminf_{n\to\infty}\min_{1\le k\le n}|x - x_{k,n}|^{1/n} < 1\}$, then T is of Lebesgue measure zero and for $x \in [-1, 1]\backslash T$,

$$(2)\quad \lim_{k\to\infty}(p_k+1)^{-1}\log|L_{p_k+1}(x)-|x|| = \tfrac{1}{2}[(1+x)\log(1+x)+(1-x)\log(1-x)].$$

Remark. Such a set T is not empty. A number x^* in T can be constructed as follows. Set $10^{\langle 1\rangle} := 10$, $10^{\langle 2\rangle} := 10^{10^{\langle 1\rangle}}, \dots, 10^{\langle k+1\rangle} = 10^{10^{\langle k\rangle}}, \dots$. Then

$$x^* = \frac{1}{10^{\langle 1\rangle}} - \frac{2}{10^{\langle 2\rangle}} + \cdots + \frac{1}{10^{\langle 2k-1\rangle}} - \frac{2}{10^{\langle 2k\rangle}} + \cdots$$

is a number in $(0, 1)$. It is easy to see that if $n_j := 10^{\langle 2j\rangle} + 1$, then

$$x_{k(n_j),n_j} = \frac{1}{10^{\langle 1\rangle}} - \frac{2}{10^{\langle 2\rangle}} + \cdots + \frac{1}{10^{\langle 2j-1\rangle}} - \frac{2}{10^{\langle 2j\rangle}}$$

and

$$0 < x^* - x_{k(n_j),n_j} < \frac{1}{10^{\langle 2j+1\rangle}}.$$

So

$$|x^* - x_{k(n_j),n_j}|^{1/(n_j-1)} < \frac{1}{10},$$

thus $x^* \in T$.

We shall assume Theorem 3 for the moment and prove Corollary 4. The proof of Theorem 3 will be given in §2.

We need the following elementary result for the proof of Corollary 4.

Lemma 5. *Let $W_n(x)$ be as in Theorem 3 and T as in Corollary 4. If x is an irrational number in $[-1, 1]\backslash T$ or a real number in $(-\infty, -1)\cup(1, \infty)$, then*

$$(3)\qquad \lim_{n\to\infty}|W_n(x)|^{1/n} = e^{-1}|1+x|^{(1+x)/2}|1-x|^{(1-x)/2}.$$

If x is a rational number in $[-1, 1]$, define

$$\Lambda_x := \{n : n = 1, 2, 3, \dots \text{ and } n + x(n-1) \text{ is an odd integer}\};$$

then

$$W_n(x) = 0 \qquad (n \in \Lambda_x)$$

and

$$(4)\quad \limsup_{n\to\infty}|W_n(x)|^{1/n} = \lim_{\substack{n\to\infty\\ n\notin\Lambda_x}}|W_n(x)|^{1/n} = e^{-1}(1+x)^{(1+x)/2}(1-x)^{(1-x)/2}.$$

This result may be well known. We include a proof for the sake of completeness. For relevant discussions, see [D, p. 84; IK, Chapter 6, §3.6].

Proof of Lemma 5. First, if x is a real number in $(-\infty, -1) \cup (1, \infty)$, then $\log|x-t|$ is a continuous function of t over the interval $[-1, 1]$ and

$$\frac{1}{n-1}\log|W_n(x)| - \frac{1}{n-1}\log|x-1| = \frac{1}{n-1}\sum_{k=1}^{n-1}\log|x - x_{k,n}|$$

is just a Riemann sum of the integral $\frac{1}{2}\int_{-1}^{1}\log|x-t|\,dt$; thus

$$\lim_{n\to\infty}\frac{1}{n-1}\log|W_n(x)| = \frac{1}{2}\int_{-1}^{1}\log|x-t|\,dt. \tag{5}$$

Since

$$\frac{1}{2}\int_{-1}^{1}\log|x-t|\,dt = \frac{1}{2}[(1+x)\log|1+x| + (1-x)\log|1-x|] - 1$$

for all $x \in \mathbf{R}$, it follows that (5) implies (3).

Next, when x is an irrational number in $[-1, 1]$, we show

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{k(n)-1}\log(x - x_{k,n}) = \frac{1}{2}\int_{-1}^{x}\log(x-t)\,dt \tag{6}$$

and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=k(n)+2}^{n}\log(x_{k,n} - x) = \frac{1}{2}\int_{x}^{1}\log(t-x)\,dt, \tag{7}$$

where $k(n) := \max\{k : k \geq 1 \text{ and } x_{k,n} < x\}$. (The dependence of $k(n)$ on x is omitted for simplicity of notation.)

We only prove (6), the proof of (7) is similar.

Let the irrational $x \in [-1, 1]$ be fixed. Since $\log(x-t)$ is a decreasing function of t for $t < x$, we have

$$\frac{1}{2}\int_{x_{k-1,n}}^{x_{k,n}}\log(x-t)\,dt < \frac{1}{n-1}\log(x - x_{k-1,n})$$

for all $k \leq k(n)$ and $k \geq 2$. Summing both sides for $k = 2, 3, \ldots, k(n)$, we get

$$\frac{1}{2}\sum_{k=2}^{k(n)}\int_{x_{k-1,n}}^{x_{k,n}}\log(x-t)\,dt < \frac{1}{n-1}\sum_{k=2}^{k(n)}\log(x - x_{k-1,n}). \tag{8}$$

Since the left side equals $\frac{1}{2}\int_{-1}^{x_{k(n),n}}\log(x-t)\,dt$ and $x_{k(n),n} \to x$ as $n \to \infty$, the existence of the improper integral $\int_{-1}^{x}\log(x-t)\,dt$ implies that the left side of (8) tends to $\frac{1}{2}\int_{-1}^{x}\log(x-t)\,dt$ as $n \to \infty$. Thus, letting $n \to \infty$ in (8), we get

$$\frac{1}{2}\int_{-1}^{x}\log(x-t)\,dt \leq \liminf_{n\to\infty}\frac{1}{n-1}\sum_{k=1}^{k(n)-1}\log(x - x_{k,n}). \tag{9}$$

On the other hand, we have

$$\frac{1}{n-1}\log(x-x_{k,n}) < \frac{1}{2}\int_{x_{k-1,n}}^{x_{k,n}} \log(x-t)\,dt.$$

Summing both sides for $k = 2, 3, \dots, k(n)-1$ and using arguments similar to those used to prove (9), we obtain

$$\limsup_{n\to\infty} \frac{1}{n-1}\sum_{k=2}^{k(n)-1} \log(x-x_{k,n}) \le \frac{1}{2}\int_{-1}^{x} \log(x-t)\,dt. \tag{10}$$

From (9) and (10), equality (6) follows.

Now, if irrational x is not in T, since we have

$$\frac{2}{n-1}\log\left(\min_{1\le k\le 1}|x-x_{k,n}|\right) \le \frac{1}{n-1}\log|(x-x_{k(n),n})(x-x_{k(n)+1,n})| \le 0,$$

then

$$\lim_{n\to\infty}\frac{1}{n-1}\log|(x-x_{k(n),n})(x-x_{k(n)+1,n})| = 0. \tag{11}$$

Applying (6), (7), and (11), we conclude (3).

We have proved the first half of the lemma. Next, we assume x is a rational number in $[-1, 1]$. If $n \in \Lambda_x$, then

$$x = x_{(n+x(n-1)+1)/2,\,n},$$

hence

$$W_n(x) = W_n(x_{(n+x(n-1)+1)/2,\,n}) = 0.$$

Thus the first equality in (4) follows if we can prove the second equality. If $n \notin \Lambda_x$, then $x \ne x_{k,n}$, $k = 1, 2, \dots, n$, $n = 1, 2, 3, \dots$. Since numbers in $T\backslash\{-1, 0, 1\}$ must be transcendental by Liouville's theorem (cf. [HW, Theorem 191, p. 161]), we also have $x \notin T$. So (11) still holds. Then, similar to the proof of (6), we can show

$$\lim_{\substack{n\to\infty\\ n\notin\Lambda_x}} \frac{1}{n}\sum_{k=1}^{k(n)-1} \log(x-x_{k,n}) = \frac{1}{2}\int_{-1}^{x}\log(x-t)\,dt$$

and

$$\lim_{\substack{n\to\infty\\ n\notin\Lambda_x}} \frac{1}{n}\sum_{k=k(n)+2}^{n} \log(x_{k,n}-x) = \frac{1}{2}\int_{x}^{1}\log(t-x)\,dt,$$

which, together with (11), imply the second equality in (4). □

Proof of Corollary 4. From equality (3), for irrational $x \in [-1, 1]\backslash T$, $\lim_{n\to\infty}|W_n(x)|^{1/n}$ exists; so $\lim_{k\to\infty}|W_{p_k+1}(x)|^{1/(p_k+1)}$ exists and is equal to $\lim_{n\to\infty}|W_n(x)|^{1/n}$, i.e.,

$$\lim_{n\to\infty}|W_n(x)|^{1/n} = \lim_{k\to\infty}|W_{p_k+1}(x)|^{1/(p_k+1)} = \frac{1}{e}(1+x)^{(1+x)/2}(1-x)^{(1-x)/2}. \tag{12}$$

Now, for rational x with $0 < |x| < 1$, say $x = p/q$ with $(p, q) = 1$ and $q > 1$, note that

$$n \in \Lambda_x \quad \text{implies} \quad (n-1)/q \text{ is an integer};$$

we conclude that $\{p_k + 1 : p_k > q\} \cap \Lambda_x = \varnothing$. Then (4) implies

$$(13)\quad \lim_{k\to\infty} |W_{p_k+1}(x)|^{1/(p_k+1)} = \lim_{\substack{n\to\infty \\ n\notin\Lambda_x}} |W_n(x)|^{1/n} = \frac{1}{e}(1+x)^{(1+x)/2}(1-x)^{(1-x)/2}.$$

Combining (12) and (13), by Theorem 3 we obtain (2).

To prove (1), in view of Theorem 3 and equations (12) and (13), it suffices to show (no matter whether $x \in T$ or not)

$$\begin{aligned}\limsup_{n\to\infty} |W_n(x)|^{1/n} &= \limsup_{k\to\infty} |W_{p_k+1}(x)|^{1/(p_k+1)} \\ &= \frac{1}{e}(1+x)^{(1+x)/2}(1-x)^{(1-x)/2}, \qquad x \in (-1, 1).\end{aligned}$$

Note that for $x \in (-1, 1)$ and n large enough,

$$\frac{1}{n}\log|W_n(x)| \le \frac{1}{n}\log\left|\frac{W_n(x)}{(x - x_{k(n),n})(x - x_{k(n)+1,n})}\right|.$$

Using (6) and (7), the left side tends to $\frac{1}{2}\int_{-1}^{1}\log|x-t|\,dt$. So letting $n \to \infty$ gives

$$\limsup_{n\to\infty} \frac{1}{n}\log|W_n(x)| \le \frac{1}{2}\int_{-1}^{1}\log|x-t|\,dt.$$

We need only to show

$$\limsup_{k\to\infty} \frac{1}{p_k+1}\log|W_{p_k+1}(x)| \ge \frac{1}{2}\int_{-1}^{1}\log|x-t|\,dt.$$

In fact, since (6) and (7) hold with $p_k + 1$ replacing n, we only have to show

$$(14)\qquad \limsup_{n=p_j+1\to\infty} \frac{1}{n}\log|(x - x_{k(n),n})(x - x_{k(n)+1,n})| = 0, \qquad |x| < 1.$$

If (14) is false, then there exists $r < 0$ such that, for j large enough,

$$(15)\qquad \frac{1}{n}\log|(x - x_{k(n),n})(x - x_{k(n)+1,n})| < r < 0, \qquad n = p_j + 1.$$

Assume $|x - x_{h(n),n}| = \min_{1\le k\le n}|x - x_{k,n}|$; then $h(n) = k(n)$ or $k(n)+1$ and

$$|(x - x_{k(n),n})(x - x_{k(n)+1,n})| \ge \frac{1}{2(n-1)}|x - x_{h(n),n}|.$$

Using this inequality in (15), we have, for some $\rho \in (0, 1)$ and j large enough,

$$|x - x_{h(n),n}| < \rho^n, \qquad n = p_j + 1.$$

So

$$(16)\qquad |x_{h(n),n} - x_{h(n'),n'}| \le |x - x_{h(n),n}| + |x - x_{h(n'),n'}| < 2\rho^n$$

with $n = p_j + 1$ and $n' = p_{j+1} + 1$. Denote $s_j = h(n) - 1$ and $s_{j+1} = h(n') - 1$. Then (16) can be written as

$$\left|\frac{2s_j}{p_j} - \frac{2s_{j+1}}{p_{j+1}}\right| < 2\rho^{p_j}$$

or, equivalently,

$$|s_j p_{j+1} - s_{j+1} p_j| < p_j p_{j+1} \rho^{p_j} \quad \text{for } j \text{ large enough.}$$

Now using the famous Bertrand's Postulate in number theory (cf. [HW, Theorem 418, p. 343]), we know $p_{j+1} < 2p_j$. So

$$|s_j p_{j+1} - s_{j+1} p_j| < 2p_j^2 \rho^{p_j} \to 0 \quad \text{as } j \to \infty.$$

Since $s_j p_{j+1} - s_{j+1} p_j$ is an integer, it follows that

$$s_j p_{j+1} - s_{j+1} p_j = 0 \quad \text{for } j \text{ large enough.}$$

But this is absurd because $0 < s_j < p_j$ and $0 < s_{j+1} < p_{j+1}$ for j large enough and both p_j and p_{j+1} are primes. So (14) must be true, thus (1) holds.

Finally, we prove T is of Lebesgue measure zero. Note that

$$T = \bigcup_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} \bigcup_{k=1}^{n} \left(x_{k,n} - \left(1 - \frac{1}{m}\right)^n, x_{k,n} + \left(1 - \frac{1}{m}\right)^n \right).$$

Let μ^* denote the Lebesgue outer measure. Then

$$\begin{aligned}
\mu^*(T) &\le \sum_{m=1}^{\infty} \mu^* \left(\overline{\lim_{n\to\infty}} \bigcup_{k=1}^{n} \left(x_{k,n} - \left(1 - \frac{1}{m}\right)^n, x_{k,n} + \left(1 - \frac{1}{m}\right)^n \right) \right) \\
&= \sum_{m=1}^{\infty} \limsup_{n\to\infty} \mu^* \left(\bigcup_{k=1}^{n} \left(x_{k,n} - \left(1 - \frac{1}{m}\right)^n, x_{k,n} + \left(1 - \frac{1}{m}\right)^n \right) \right) \\
&\le \sum_{m=1}^{\infty} \limsup_{n\to\infty} 2n \left(1 - \frac{1}{m}\right)^n = 0.
\end{aligned}$$

This completes the proof of Corollary 4. □

Before we prove Theorem 3, we would like to remark that a closer look at the proof of Theorem 1 given in [BMS] would suggest a possible proof of Theorem 3, but their proof uses Lagrange interpolation formula and involves a tricky transformation and hypergeometric series identities, which is entirely different from Bernstein's approach of using Newton's interpolation formula. As another goal of this note we present a short and elementary proof of Theorem 3 by using Bernstein's approach.

Recall that the Lagrange interpolation polynomial $L_n(f; x)$ can be expressed by Newton's formula (cf. [D, N])

$$L_n(f; x) = \sum_{k=0}^{n-1} \left(\frac{n-1}{2}\right)^k \frac{\Delta_n^k f(-1)}{k!} (x - x_{1,n}) \cdots (x - x_{k,n}),$$

where

$$\Delta_n^l f(-1) = \sum_{r=0}^{l} (-1)^{l-r} \binom{l}{r} f(x_{r+1,n}), \qquad l = 0, 1, \ldots, n-1.$$

2. Proof of Theorem 3

We prove the theorem for x such that $x < 0$; the result when $0 < x$ can then be obtained by symmetry. Denote $n' := [(n-1)/2]$, where $[t]$ denotes the largest integer $\le t$. If

$$\phi(t) = \begin{cases} 0 & \text{for } t \le 0, \\ t & \text{for } 0 \le t, \end{cases}$$

then

$$\frac{L_n(x)-|x|}{2}=\frac{L_n(x)+x}{2}=L_n(\phi;x)$$

since $x=L_n(t;x)$. Now from Newton's formula we can get

(17)

$$L_n(\phi;x)=\sum_{m=1}^{n'}(-1)^{m-1}\left(\frac{n-1}{2}\right)^{n'+m}\times\frac{[1+(n-1-2n')(n'-m)](x-x_1)(x-x_2)\cdots(x-x_{n'+m})}{(m-1)!n'!(n'+m)(n'+m-1)}.$$

In fact, when n is odd, (13) is established as formula (70) of [N, p. 31]. The proof of (17) when n is even is entirely analogous. Next, write (17) as

$$L_n(\phi;x)=(-1)\frac{(x-x_1)\cdots(x-x_{n'})}{n'!}\times\sum_{m=1}^{n'}\left(\frac{n-1}{2}\right)^{n'+m}\times\frac{[1+(n-1-2n')(n'-m)](x_{n'+1}-x)\cdots(x_{n'+m}-x)}{(m-1)!(n'+m)(n'+m-1)};$$

then every term in the sum is positive. So

$$(18)\quad |L_n(\phi;x)|\geq\frac{|(x-x_1)\cdots(x-x_{n'})|}{n'!}\cdot\left(\frac{n-1}{2}\right)^{2n'}\frac{(x_{n'+1}-x)\cdots(x_{2n'}-x)}{(n'-1)!2n'(2n'-1)}$$

and

(19)

$$\begin{aligned}|L_n(\phi;x)|&\leq\frac{|(x-x_1)\cdots(x-x_{n'})|}{n'!}\times\sum_{m=1}^{n'}\left(\frac{n-1}{2}\right)^{n'+m}\times\frac{n'(x_{n'+1}-x)\cdots(x_{n'+m}-x)(x_{n'+m+1}-x)\cdots(x_{2n'}-x)}{(m-1)!(n'+m)(n'+m-1)\cdot x_{n'+m+1}\cdots x_{2n'}}\\&\leq\frac{|(x-x_1)\cdots(x-x_{n'})|}{n'!}\times\sum_{m=1}^{n'}\left(\frac{n-1}{2}\right)^{n'+m}\times\frac{n'(x_{n'+1}-x)\cdots(x_{2n'}-x)}{(m-1)!(n'+1)(n')\cdot(\frac{2}{n-1})^{n'-m}(m-\frac{1}{2})(m)(m+1)\cdots(n'-1)}\\&\leq\frac{|(x-x_1)\cdots(x-x_{n'})|}{n'!}\cdot n'\cdot\left(\frac{n-1}{2}\right)^{2n'}\frac{n'(x_{n'+1}-x)\cdots(x_{2n'}-x)}{(n'+1)n'\cdot(n'-1)!(1-\frac{1}{2})}\\&\leq\frac{2|(x-x_1)\cdots(x-x_{n'})|}{(n'!)^2}\cdot n'\cdot\left(\frac{n-1}{2}\right)^{2n'}(x_{n'+1}-x)\cdots(x_{2n'}-x).\end{aligned}$$

Hence, from (18) and (19),

$$\lim_{n\to\infty}\left|\frac{L_n(\phi;x)}{W_n(x)}\right|^{1/n} = \lim_{n\to\infty}\left\{\frac{1}{(n'!)^2}\left(\frac{n-1}{2}\right)^{2n'}\right\}^{1/n}. \tag{20}$$

Finally, by using Stirling's formula

$$n! = \sqrt{2\pi n}\,n^n e^{-n}(1+w_n), \qquad \lim w_n = 0,$$

we can easily verify that the limit in (20) is e. □

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