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THE POINCARÉ INEQUALITY AND ENTIRE FUNCTIONS

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ABSTRACT. Inequalities for spaces of entire functions on $\mathbf{C}^n$, which generalize the Poincaré inequality for Gaussian measure, are obtained. The relationship between these inequalities and hypercontractive estimates for diffusion semigroups are discussed.

1. INTRODUCTION

The Poincaré inequality for $\mathbf{R}^n$ is

$$\int_{\mathbf{R}^n} |f|^2\, d\mu - \left(\int_{\mathbf{R}^n} f\, d\mu\right)^2 \le \int_{\mathbf{R}^n} |\nabla f|^2\, d\mu, \tag{1.1}$$

where $d\mu$ is normalized Gaussian measure on $\mathbf{R}^n$,

$$d\mu = d\mu(x) = (2\pi)^{-n/2} e^{-|x|^2/2}\, dx.$$

In a recent paper, Beckner [Be2] discusses the role of this inequality in applications, especially in regards to quantum mechanics and the uncertainty principle, and obtains

Theorem 1 (Beckner). *For $f \in L^2(d\mu)$, $1 \le p \le 2$, and $e^{-t} = \sqrt{p-1}$,*

$$\int_{\mathbf{R}^n} |f|^2\, d\mu - \int_{\mathbf{R}^n} |e^{-tN} f|^2\, d\mu \le (2-p) \int_{\mathbf{R}^n} |\nabla f|^2\, d\mu \tag{1.2}$$

and

$$\int_{\mathbf{R}^n} |f|^2\, d\mu - \left(\int_{\mathbf{R}^n} |f|^p\, d\mu\right)^{2/p} \le (2-p) \int_{\mathbf{R}^n} |\nabla f|^2\, d\mu. \tag{1.3}$$

Inequalities (1.2) and (1.3) can be viewed as interpolating inequalities for the Poincaré inequality (1.1) and the logarithmic Sobolev inequality of Gross [G]

$$\int_{\mathbf{R}^n} |f|^2 \ln|f|\, dg - \int_{\mathbf{R}^n} |f|^2\, dg \ln\left(\int_{\mathbf{R}^n} |f|^2\, dg\right)^{1/2} \le \int_{\mathbf{R}^n} |\nabla f|^2\, dg. \tag{1.4}$$

Integrating by parts on the right-hand side of (1.1) yields

$$\int_{\mathbf{R}^n} |f|^2\, d\mu - \left(\int_{\mathbf{R}^n} f\, d\mu\right)^2 \le \int_{\mathbf{R}^n} (Nf) f\, d\mu, \tag{1.1$'$}$$

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where the number operator $N = -\Delta + x \cdot \nabla$ is the generator of the Hermite semigroup on $\mathbf{R}^n$, also called the Ornstein-Uhlenbeck process. Inequality (1.4) corresponds to Nelson's hypercontractive estimate for the Hermite semigroup [N]: for $1 \leq p \leq 2$,

$$e^{-t} \leq \sqrt{p-1} \Rightarrow \int_{\mathbf{R}^n} |e^{-tN} f|^2 \, d\mu \leq \left(\int_{\mathbf{R}^n} |f|^p \, d\mu \right)^{2/p}. \tag{1.5}$$

Beckner observed that inequality (1.3) follows from (1.2) by using (1.5) and that inequality (1.4) can be recovered from (1.3) by a limiting argument as p approaches 2.

In this note, we will obtain analogues of inequalities (1.2) and (1.3) for the spaces $A^p = A^p(\mathbf{C}^n)$, $0 < p < \infty$, consisting of entire functions Φ on $\mathbf{C}^n$ such that

$$\int_{\mathbf{C}^n} |\Phi(z)|^p \, d\mu < \infty,$$

where for $z = x + iy$, $x, y \in \mathbf{R}^n$,

$$d\mu = d\mu(z) = (2\pi)^{-n} e^{-|z|^2/2} \, d|z| = d\mu(x) \, d\mu(y)$$

denotes normalized Gaussian measure on $\mathbf{C}^n$. The Poincaré inequality in this setting is

$$\int_{\mathbf{C}^n} |f|^2 \, d\mu - (f(0))^2 \leq 2 \int_{\mathbf{C}^n} \left| \frac{df}{dz} \right|^2 d\mu. \tag{1.6}$$

In §2, we will prove

Theorem 2. *For $f \in A^2(\mathbf{C}^n)$ and $|\omega| \leq 1$,*

$$\int_{\mathbf{C}^n} |f(z)|^2 \, d\mu - \int_{\mathbf{C}^n} |f(\omega z)|^2 \, d\mu \leq 2(1 - |\omega|^2) \int_{\mathbf{C}^n} \left| \frac{df}{dz} \right|^2 d\mu, \tag{1.7}$$

and, for $0 < p \leq 2$,

$$\int_{\mathbf{C}^n} |f(z)|^2 \, d\mu - \left(\int_{\mathbf{C}^n} |f(z)|^p \, d\mu \right)^{2/p} \leq (2-p) \int_{\mathbf{C}^n} \left| \frac{df}{dz} \right|^2 d\mu. \tag{1.8}$$

Inequality (1.6) can be recovered from (1.7) by taking the limit as $|\omega| \to 0$.

Inequalities (1.7) and (1.8) interpolate between the Poincaré inequality (1.6) and the logarithmic inequality

$$\int_{\mathbf{C}^n} |f|^2 \ln|f| \, d\mu - \int_{\mathbf{C}^n} |f|^2 \, d\mu \ln \left(\int_{\mathbf{C}^n} |f|^2 \, d\mu \right)^{1/2} \leq \int_{\mathbf{C}^n} \left| \frac{df}{dz} \right|^2 d\mu. \tag{1.9}$$

Inequality (1.9) corresponds to the following hypercontractive estimate: for $0 < p \leq 2$,

$$|\omega| \leq \sqrt{\frac{p}{2}} \Rightarrow \int_{\mathbf{C}^n} |f(\omega z)|^2 \, d\mu \leq \left(\int_{\mathbf{C}^n} |f(z)|^p \, d\mu \right)^{2/p}. \tag{1.10}$$

Both (1.9) and (1.10) are due to Janson [J].

In this case, note that (1.8) follows from (1.7) by setting $\omega = \sqrt{p/2}$ and using (1.10). We can then use a limiting argument as p approaches 2 to recover inequality (1.9) from (1.8).

2. Proof of Theorem 2

From the preceding remarks, it will be enough to show that (1.7) holds. To obtain (1.7) we will need to recall some properties of A^2. The space A^2 arises in the study of canonical operators in quantum mechanics through the ideas of Fock, Dirac, Bargmann [Ba], and Segal [S1, S2]. The properties described below are worked out in detail in [Ba].

On A^2 the family of polynomials

$$f_\alpha(z) = \frac{z^\alpha}{2^{(|\alpha|+n)/4}\sqrt{\alpha!}} = \prod_{k=1}^{n} \frac{z_k^{\alpha_k}}{2^{(\alpha_k+1)/4}\sqrt{\alpha_k!}},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index of nonnegative integers, is an orthonormal basis with respect to the inner product

$$\langle f, g\rangle = \int_{\mathbf{C}^n} \overline{f(z)}g(z)\,d\mu(z).$$

The operator (of multiplication by) z_k is the adjoint of $2\partial/\partial z_k = \partial/\partial x_k - i\partial/\partial y_k$, and the analogue of the number operator in this setting is $N = z\cdot d/dz$. N is a positive, selfadjoint operator, which acts on the basis according to the rule

$$Nf_\alpha = |\alpha| f. \tag{2.1}$$

One can also check that

$$2\int_{\mathbf{C}^n} \left|\frac{df}{dz}\right|^2 d\mu = \int_{\mathbf{C}^n} \overline{(Nf)} f\,d\mu. \tag{2.2}$$

It is easy to see that N generates the semigroup $P(t)$, defined for $t \geq 0$ by

$$(P(t)f)(z) = (e^{-tN}f)(z) = f(e^{-t}z)$$

or, setting $e^{-t} = \omega$,

$$(P(t)f)(z) = f(\omega z).$$

This makes the relationship between Theorems 1 and 2 clear. In fact, both inequalities (1.2) and (1.7) can be written in the form

$$\int |f|^2\,d\mu - \int |e^{-tN}f|^2\,d\mu \leq (1 - e^{-2t})\langle Nf, f\rangle.$$

The differences in Theorems 1 and 2 are primarily a result of the stronger hypercontractivity in the complex case and the fact that, since $d\mu(z)$ is invariant under complex rotations, we need not restrict ω to be real. We will now show inequality (1.7) to be a consequence of spectral considerations. To see this, let π_k be the projection map from A^2 to the subspace spanned by the f_α's with length $|\alpha| = k$. Then using (2.1) and (2.2), inequality (1.7) is equivalent to

$$\int_{\mathbf{C}^n} |\pi_k f|^2\,d\mu - |\omega|^{2k}\int_{\mathbf{C}^n} |\pi_k f|^2\,d\mu \leq (1 - |\omega|^2)k\int_{\mathbf{C}^n} |\pi_k f|^2\,d\mu. \tag{2.3}$$

Inequality (2.3) is determined by the relation

$$1-|\omega|^{2k}\leq(1-|\omega|^2)k\,,$$

which is a consequence of the identity

$$1-|\omega|^{2k}=(1-|\omega|^2)(1+|\omega|^2+|\omega|^4+\cdots+|\omega|^{2(k-1)})$$

and the fact that, with $|\omega|\leq 1$, $(1+|\omega|^2+|\omega|^4+\cdots+|\omega|^{2(k-1)})\leq k$.

Since $|\omega|\leq 1$ and $k\geq 1$ implies $1\leq(1-|\omega|^{2k})/(1-|\omega|^2)$, inequality (1.7) gives the following smooth interpolation for the Poincaré inequality (1.6):

$$\begin{aligned}\int_{\mathbf{C}^n}|f(z)|^2\,d\mu-(f(0))^2&\leq\left(\frac{\int_{\mathbf{C}^n}|f(z)|^2\,d\mu-\int_{\mathbf{C}^n}|f(\omega z)|^2\,d\mu}{1-|\omega|^2}\right)\\&\leq 2\int_{\mathbf{C}^n}\left|\frac{df}{dz}\right|^2\,d\mu.\end{aligned}\tag{2.4}$$

3. Further results

Results similar to Theorems 1 and 2 can be obtained for the sphere S^n by using the technique of §2 and either the hypercontractive estimates for the heat semigroup due to Mueller and Weissler [MW] or those for the Poisson semigroup due to Beckner [Be1]. In fact, this was carried out for the Poisson semigroup by Beckner [Be2], who found that, for $1\leq p\leq 2$,

$$\int_{S^n}|F|^2\,d\xi-\left(\int_{S^n}|F|^p\,d\xi\right)^{2/p}\leq\frac{2-p}{n}\int_{S^n}|\nabla F|^2\,d\xi\,,\tag{3.1}$$

where $d\xi$ denotes normalized surface measure on S^n.

Let P^2 denote 2-dimensional real projective space, which we view as S^2 with antipodal points identified. We again use $d\xi$ to denote normalized surface measure. Then using (3.1) and the method of proof in the author's paper [P], we obtain

Theorem 3. *For $F\in L^2(P^2)$ and $1\leq p\leq 2$,*

$$\int_{P^2}|F|^2\,d\xi-\left(\int_{P^2}|F|^p\,d\xi\right)^{2/p}\leq\frac{2-p}{4}\int_{P^2}|\nabla F|^2\,d\xi.\tag{3.2}$$

If we divide both sides of (3.2) by $(2-p)$ and take the limit as p approaches 2, we get the logarithmic inequality

$$\int_{P^2}|F|^2\ln|F|\,d\xi-\left(\int_{P^2}|F|^2\,d\xi\right)\ln\left(\int_{P^2}|F|^2\,d\xi\right)^{1/2}\leq\frac{1}{4}\int_{P^2}|\nabla F|^2\,d\xi.\tag{3.3}$$

Inequality (3.3) corresponds to the hypercontractive estimate for the heat semigroup on P^2: for $1\leq p\leq 2$,

$$e^{-4t}\leq\sqrt{p-1}\Rightarrow\int_{P^2}|e^{t\Delta}F|^2\,d\xi\leq\left(\int_{P^2}|F|^p\,d\xi\right)^{2/p}.\tag{3.4}$$

Both (3.3) and (3.4) appear in [P].

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