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A GENERAL HOPF LEMMA AND PROPER HOLOMORPHIC MAPPINGS BETWEEN CONVEX DOMAINS IN $\mathbb{C}^n$

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ABSTRACT. We use a general version of the well-known Hopf lemma to study boundary regularity of proper holomorphic mappings between some bounded convex domains in $\mathbb{C}^n$ which carry no boundary regularity assumption.

0. INTRODUCTION

Let Ω_1 and Ω_2 be domains in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. A continuous mapping $f: \Omega_1 \to \Omega_2$ is proper provided $f^{-1}(K)$ is compact in Ω_1 whenever K is compact in Ω_2. If Ω_1 and Ω_2 are bounded, this is equivalent to the requirement that $f(z_j) \to \partial\Omega_2$ whenever $\{z_j\} \subset \Omega_1$ is such that $z_j \to \partial\Omega_1$. A biholomorphic mapping is proper since in this case f^{-1} is continuous.

The problem of boundary regularity of proper holomorphic mappings has been studied by many authors (see the survey article [F] and the references therein). In most cases the domains in question are assumed to possess at least C^2 boundary regularity. This paper studies the problem for certain bounded domains in $\mathbb{C}^n$ which carry no such assumption.

In §1 we fix notation and recall some fundamental ideas, including a generalization of the well-known Hopf lemma which requires only a cone condition on the domain in question rather than boundary smoothness. In §2 we apply this result to obtain some sufficient conditions on bounded domains $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ for a proper holomorphic mapping $f: \Omega_1 \to \Omega_2$ to have a Hölder continuous extension to $\overline{\Omega}_1$. In particular, we study a case where Ω_1 and Ω_2 are convex with no presupposed boundary regularity.

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1. PRELIMINARIES

We recall some important notions to be used in the sequel. Ω denotes a domain (= connected open set) and B_n denotes the unit ball in $\mathbb{C}^n$ defined

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via the usual (Hermitian) inner product. If $n = 1$, write $B_n = \Delta$, the unit disk in $\mathbb{C}$.

Definition 1.1. Let $\Omega \subset \mathbb{C}^n$. The Kobayashi metric $\kappa_\Omega \colon T(\Omega) \to \mathbb{R}^+$ is given by

$$\kappa_\Omega(z\,;v) = \inf\{|u| : \exists f \in \mathrm{Hol}(\Delta, \Omega) \text{ such that } f(0) = z,\ f'(0)u = v\}.$$

General properties of κ_Ω may be found, for example, in [K] or [Kr2].

If $\Omega \Subset \mathbb{C}^n$ is convex, $z \in \Omega$, and $v \in \mathbb{C}^n$, denote by $r_\Omega(z\,;v)$ the radius of the largest one complex-dimensional closed disk, centred at z, tangent to v, and contained in $\overline{\Omega}$. In this case, Graham [G2, G3] showed that for any $v \in \mathbb{C}^n$ we have

$$\frac{|v|}{2r_\Omega(z\,;v)} \le \kappa_\Omega(z\,;v) \le \frac{|v|}{r_\Omega(z\,;v)} \quad \forall z \in \Omega. \tag{1}$$

Let $\Omega \subset \mathbb{C}^n$. Recall that an upper semicontinuous function $\varphi \colon \Omega \to [-\infty, \infty)$ is plurisubharmonic (plush) if for every $z, w \in \mathbb{C}^n$, the function $\lambda \to \varphi(\lambda z + w)$ is subharmonic on $\Omega_{zw} = \{\lambda \in \mathbb{C} : \lambda z + w \in \Omega\}$. A pluripolar set is the $-\infty$ set of a nontrivial plush function.

We state a theorem, which gathers several important results about proper holomorphic mappings. These results appear in [Ru, Chapter 15] and we adopt the notation used there. Let $\Omega_1, \Omega_2 \subset \mathbb{C}^n$, and let $f \colon \Omega_1 \to \Omega_2$ be proper holomorphic. Let $E = \{z \in \Omega \colon \det[f'(z)] = 0\}$. For any set A, denote by $\#(A)$ the cardinality of A. In this situation we have

Theorem 1.2. (i) $f(\Omega_1) = \Omega_2$.

(ii) $\Omega_2 \backslash f(E)$ *is open, connected, and dense in* Ω_2.

(iii) $f(E)$ *is an analytic subvariety of* Ω_2.

(iv) *There is a positive integer* m *(called the multiplicity of* f*) such that*:

(a) *If* $w \in f(E)$ *then* $\#(f^{-1}(w)) < m$.

(b) *If* $w \in \Omega_2 \backslash f(E)$ *then* $\#(f^{-1}(w)) = m$ *and there is a neighbourhood* W *of* w *and* m *disjoint open connected sets* $U_1, \dots, U_m \subset \Omega_1$ *such that* $f^{-1}(W) = U_1 \cup \cdots \cup U_m$ *and* $f_j = f|_{U_j}$ *is biholomorphic on* U_j *with* $f \circ f_j^{-1}(w) = w$, $1 \le j \le m$.

We fix some further notation. For $z \in \Omega \subset \mathbb{C}^n$, denote by $d_\Omega(z)$ the Euclidean distance from z to $\partial\Omega$. For $p \in \mathbb{C}^n$, $\theta \in (0, \pi)$, $v \in \partial B_n$ (considered as a unit vector), and $r > 0$, denote by $\Gamma(p, \theta, v, r)$ the open cone in $\mathbb{C}^n$ with vertex p, aperture θ, axis along v, and height r. To be explicit, set $H = \{z \in \mathbb{C}^n : \mathrm{Re}\langle z, v\rangle = 0\}$; H is the $(2n-1)$ real-dimensional boundary of a half space $\sqcap$, which has v as an inner unit normal vector. Thus

$$\Gamma(p, \theta, v, r) = \{z \in \sqcap + p : |z - p| < a d_{\sqcap + p}(z),\ |z - p| < r\},$$

where $a > 1$ is given by $\theta = 2\cos^{-1}(1/a)$. The axis of $\Gamma(p, \theta, v, r)$ is the segment $\{p + tv : 0 < t < r\}$.

Definition 1.3. Let $\Omega \subset \mathbb{C}^n$ and let $\theta \in (0, \pi)$. We say that Ω satisfies a θ-cone condition if there is an $r > 0$ with the following property: Each $z \in \Omega$ sufficiently close to $\partial\Omega$ lies on the axis of a cone $\Gamma(p, \theta, v, r) \subset \Omega$ for some $p \in \partial\Omega$, $v \in \partial B_n$.

Such a condition arises in potential theory and the theory of partial differential equations. For example, a Lipschitz domain (a domain whose interior and boundary are given locally by a Lipschitz function) satisfies a θ-cone condition.

The following is the promised version of the Hopf lemma, the proof of which is a modification of that of [FS, Proposition 12.2]. We are grateful to the referee for bringing to our attention that an even more general version is known [O, Mi1, Mi2].

Proposition 1.4. *Let $\Omega \Subset \mathbb{C}^n$ satisfy a θ-cone condition. Let $\varphi\colon \Omega \to [-\infty, 0)$ be plush. There is a $c > 0$ and an $\alpha > 1$ $(\alpha = \pi/\theta)$ such that*

$$\varphi(z) \leq -cd_\Omega^\alpha(z) \quad \forall z \in \Omega.$$

Remark 1.4.1. If $\Omega \Subset \mathbb{C}^n$ is convex then Ω satisfies a θ-cone condition (see the proof of Lemma 2.2). The integrated form k_Ω of κ_Ω is the well-known Kobayashi distance on Ω [Roy] (see also [K, Kr2]). We remark further that in this case Lempert [L] showed that for each fixed $z_0 \in \Omega$ the function $\log\tanh k_\Omega(z_0, \cdot)$ is plush on Ω. Now whenever $\varepsilon > 0$ is small, we have $-x < \log(1-(1-\varepsilon)x)$ for small $x > 0$. Proposition 1.4 together with Lempert's result implies then that there is a $c > 0$ (depending only on z_0) and an $\alpha > 1$ such that

$$k_\Omega(z_0, z) \leq c - \tfrac{1}{2}\log d_\Omega^\alpha(z) \quad \forall z \in \Omega.$$

This inequality appears in [Me].

Remark 1.4.2. If $\Omega \Subset \mathbb{C}^n$ has piecewise smooth boundary in the sense of [R1] then Ω satisfies a θ-cone condition. Clearly, such a domain need not be convex. Conversely, a (bounded) convex domain need not have piecewise smooth boundary. See also Remark 2.6.1.

2. Application to proper holomorphic mappings

Definition 2.1. Let $\Omega \Subset \mathbb{C}^n$ be starshaped with respect to $0 \in \Omega$. The Minkowski Functional $\mu_\Omega : \mathbb{C}^n \to \mathbb{R}$ for Ω with respect to 0 is given by

$$\mu_\Omega(z) = \begin{cases} \inf[t > 0 : z/t \in \Omega], & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Then we have $\Omega = \{\mu_\Omega < 1\}$, $\partial\Omega = \{\mu_\Omega = 1\}$, and $\Omega^c = \{\mu_\Omega > 1\}$.

Lemma 2.2. *Let $\Omega \Subset \mathbb{C}^n$ be convex with $0 \in \Omega$. The function $\mu_\Omega - 1 : \Omega \to [-1, 0)$ is plush and there is a $c > 0$ such that*

$$-cd_\Omega(z) \leq \mu_\Omega(z) - 1 \quad \forall z \in \Omega.$$

Proof. Since Ω is convex, μ_Ω is a convex function and the first assertion follows.

There is a $\theta \in (0, \pi)$ and an $r > 0$ such that for each $p \in \partial\Omega$ we have $\Gamma_p = \Gamma(p, \theta, -p/|p|, r) \subset \Omega$. Now to prove the second assertion it suffices to consider points $z \in \Omega$ near $\partial\Omega$. For such a z, set $p = p(z) = z/\mu_\Omega(z) \in \partial\Omega$. We may assume that $z \in \Gamma_p$. Let $a = \inf[|p| : p \in \partial\Omega]$. Then

$$\begin{aligned} d_\Omega(z) &\geq d_{\Gamma_p}(z) = |p - z|\sin(\theta/2) \\ &= (1 - \mu_\Omega(z))|p|\sin(\theta/2) \geq (1 - \mu_\Omega(z))a\sin(\theta/2), \end{aligned}$$

and the result follows. $\square$

Proposition 2.3. *Let* $\Omega_1, \Omega_2 \Subset \mathbb{C}^n$ *be convex. Let* $f: \Omega_1 \to \Omega_2$ *be proper holomorphic. There is an* $\alpha > 1$ *and constants* $a_1, a_2 > 0$ *such that*

$$a_1 d^{\alpha}_{\Omega_1}(z) \le d_{\Omega_2}(f(z)) \le a_2 d^{1/\alpha}_{\Omega_1}(z) \quad \forall z \in \Omega_1 . \tag{2}$$

Proof. We may assume that $0 \in \Omega_2$; set $\varphi_2 = \mu_{\Omega_2} - 1$. f is holomorphic and φ_2 is plush (Lemma 2.2), so $\varphi_2 \circ f$ is plush on Ω_1. Now Ω_1 satisfies a θ-cone condition, so Proposition 1.4 provides a $\gamma > 1$ and a $c > 0$ such that

$$\varphi_2 \circ f(z) \le -c d^{\gamma}_{\Omega_1}(z) \quad \forall z \in \Omega_1 .$$

The left-hand inequality in (2) now follows from Lemma 2.2.

To prove the right-hand inequality we adopt the terminology of Theorem 1.2 and employ some ideas appearing in [P]. We may assume that $0 \in \Omega_1$; set $\varphi_1 = \mu_{\Omega_1} - 1$. Fix a point $w_0 \in \Omega_2 \backslash f(E)$ and let W be a neighbourhood of w_0 as in Theorem 1.2(iv)(b). Define $\psi_j: W \to U_j$ by

$$\psi_j(w) = \varphi_1 \circ f_j^{-1}(w), \qquad 1 \le j \le m . \tag{3}$$

The function $\psi(w) = \max[\psi_j(w) : 1 \le j \le m]$ is then well defined, plush on $\Omega_2 \backslash f(E)$, and also bounded there. Now by Theorem 1.2(iii), $f(E)$ is an analytic subvariety of Ω_2 and as such it is a pluripolar set. The appropriate extension theorem (e.g., [LG, Proposition I.22]) shows that ψ extends to a plush bounded function on all of Ω_2, which we denote again by ψ.

By Proposition 1.4 there is a $\beta > 1$ and a $c_1 > 0$ such that

$$\psi(w) \le -c_1 d^{\beta}_{\Omega_2}(w) \quad \forall w \in \Omega_2 ;$$

thus,

$$\psi_j(w) \le -c_1 d^{\beta}_{\Omega_2}(w) \quad \forall w \in \Omega_2 \backslash f(E), \ 1 \le j \le m . \tag{4}$$

By Lemma 2.2 there is a $c_2 > 0$ such that

$$-c_2 d_{\Omega_1}(z) \le \varphi_1(z) \quad \forall z \in \Omega_1 ;$$

thus (observing Theorem 1.2(i)),

$$-c_2 d_{\Omega_1}(f_j^{-1}(w)) \le \varphi_1 \circ f_j^{-1}(w) \quad \forall w \in \Omega_2 \backslash f(E), \ 1 \le j \le m . \tag{5}$$

With $w = f(z)$, (3)–(5) provide a $c_3 > 0$ such that

$$d^{\beta}_{\Omega_2}(f(z)) \le c_3 d_{\Omega_1}(f_j^{-1} \circ f(z)) \quad \forall z \in \Omega_1 \backslash E, \ 1 \le j \le m .$$

Choosing the correct j, we have

$$d^{\beta}_{\Omega_2}(f(z)) \le c_3 d_{\Omega_1}(z) \quad \forall z \in \Omega_1 \backslash E . \tag{6}$$

Finally, by continuity and Theorem 1.2(ii), (6) holds for all $z \in \Omega_1$. Thus the right-hand inequality in (2) holds, and the proof is complete upon letting $\alpha = \max(\gamma, \beta)$. $\square$

Definition 2.4. Let $\Omega \Subset \mathbb{C}^n$ be convex. We say that Ω is m-convex if there is an $m \in (0, \infty)$ and a $c > 0$ such that for every $v \in \mathbb{C}^n$ we have

$$r_{\Omega}(z ; v) \le c d^{1/m}_{\Omega}(z) \quad \forall z \in \Omega .$$

We remark that a C^2-bounded domain with positive definite real Hessian is 2-convex. In general (for $n \ge 2$) we must have $m \ge 2$. m-convex domains are the focus of much of [Me]. Consideration of (1) shows that if Ω is m-convex then there is a $c > 0$ such that

$$\frac{|v|}{cd_\Omega^{1/m}(z)} \le \kappa_\Omega(z\,;v) \le \frac{|v|}{d_\Omega(z)} \quad \forall (z\,,v) \in T(\Omega)\,. \tag{7}$$

Lemma 2.5. *Let $\Omega \Subset \mathbb{C}^n$ be convex. Let $\beta \in (0\,,1)$ and $f \in C^1(\Omega)$. Suppose that there is a $c > 0$ such that $|\nabla f(z)| \le cd_\Omega^{\beta-1}(z)$ $\forall z \in \Omega$. There is a $c_1 > 0$ such that*

$$|f(z) - f(w)| \le c_1|z - w|^\beta \quad \forall z\,, w \in \Omega\,. \tag{8}$$

As such, f extends to a continuous function on $\overline{\Omega}$ and (8) *holds there also* (*i.e., the extension is Hölder continuous with exponent β*).

Proof. The first assertion follows from appropriate modifications of standard techniques that appear, for example, in [Kr1, Lemma 4.7]. In that lemma Ω has C^2 boundary only; the absence of such an assumption in the present case is made up for by the convexity hypothesis. The rest of the lemma follows from elementary arguments. □

Proposition 2.6. *Let $\Omega_1\,, \Omega_2 \Subset \mathbb{C}^n$ with Ω_1 convex and Ω_2 m-convex. Let $f\colon \Omega_1 \to \Omega_2$ be proper holomorphic. Then f extends to a Hölder continuous mapping on $\overline{\Omega}_1$.*

Proof. By the distance decreasing property of κ_Ω, (7), and Proposition 2.3 there is a $c > 0$ and an $\alpha > 1$ such that

$$|f'(z)v| \le \frac{cd_{\Omega_2}^{1/m}(f(z))|v|}{d_{\Omega_1}(z)} \le cd_{\Omega_1}^{1/\alpha m-1}(z)|v| \quad \forall (z\,,v) \in T(\Omega)\,.$$

Thus each component of f satisfies the hypothesis of Lemma 2.5 with $\beta = 1/\alpha m$, and we are done. □

Remark 2.6.1. [R1] (respectively [R2]) contains results analogous to Propositions 2.3 and 2.6 in case Ω_1 and Ω_2 are bounded domains with piecewise smooth strictly pseudoconvex boundaries (respectively, bounded convex domains with real analytic boundaries) and $f\colon \Omega_1 \to \Omega_2$ is biholomorphic rather than just proper holomorphic. Berteloot [B] has independently studied Hölder continuity for proper holomorphic maps between certain pseudoconvex domains with piecewise smooth boundaries. See also Remark 1.4.2.

We have already noted that Lemma 2.5 holds if $\Omega \Subset \mathbb{C}^n$ is C^2-bounded rather than convex [Kr1, Lemma 4.7]. Also, estimates such as (2) and (7) are already known to hold in situations where the domains in question have good boundary regularity. For example, (2) holds if $\Omega_1\,, \Omega_2 \Subset \mathbb{C}^n$ are C^∞-pseudoconvex [R2], or if $\Omega_1\,, \Omega_2 \Subset \mathbb{C}^n$ are C^2-strictly pseudoconvex; here $\alpha = 1$ [P]. Estimate (7) holds if $\Omega \Subset \mathbb{C}^n$ is pseudoconvex with real analytic boundary [DF], or if $\Omega \Subset \mathbb{C}^n$ is C^2-strictly pseudoconvex; here $m = 2$ [G1] (see also [H]). Consequently, results analogous to Proposition 2.6 hold for such cases [R2, P, H, DF]. Finally, assumptions on Ω_1 and Ω_2 may be varied considerably among these cases to obtain still more versions of Proposition 2.6.

References

[B] F. Berteloot, *Hölder continuity of proper holomorphic mappings*, Studia Math. **100** (1991), 229–235.

[DF] K. Diederich and J. E. Fornaess, *Proper holomorphic maps onto pseudoconvex domains with real analytic boundary*, Ann. of Math. (2) **110** (1979), 575–592.

[FS] J. E. Fornaess and B. Stensones, *Lectures on counterexamples in several complex variables*, Math. Notes, vol. 33, Princeton Univ. Press, Princeton, NJ, 1987.

[F] F. Forstnerič, *Proper holomorphic mappings: a survey*, Preprint Series, Dept. Math. University E. K. Ljubljana **27** (1989), 5–47.

[G1] I. Graham, *Boundary regularity of the Caratheodory and Kobayashi metrics on strongly pseudoconvex domains in* $\mathbb{C}^n$ *with smooth boundary*, Trans. Amer. Math. Soc. **207** (1975), 219–240.

[G2] ——, *Distortion theorems for holomorphic maps between convex domains in* $\mathbb{C}^n$, Complex Variables Theory Appl. **15** (1990), 37–42.

[G3] ——, *Sharp constants for the Koebe theorem and for estimates of intrinsic metrics on convex domains*, Proc. Sympos. Pure Math., vol. 52, part 2, Amer. Math. Soc., Providence, RI, 1991, pp. 233–238.

[H] G. M. Henkin, *An analytic polyhedron is not biholomorphically equivalent to a strictly pseudoconvex domain*, Soviet Math. Dokl. **14** (1973), 858–862.

[K] S. Kobayashi, *Intrinsic distances, measures and geometric function theory*, Bull. Amer. Math. Soc. **82** (1976), 357–416.

[Kr1] S. G. Krantz, *Optimal Lipschitz and* L^p *regularity for the equation* $\bar{\partial} u = f$ *on strongly pseudo-convex domains*, Math. Ann. **219** (1976), 233–260.

[Kr2] ——, *Function theory of several complex variables*, Wiley, New York, 1982.

[LG] P. Lelong and L. Gruman, *Entire functions of several complex variables*, Grundlehren Math. Wiss., vol. 282, Springer-Verlag, Berlin, 1986.

[L] L. Lempert, *La métrique de Kobayashi et la répresentation des domains sur la boule*, Bull. Soc. Math. France **109** (1981), 427–474.

[Me] P. R. Mercer, *Complex geodesics and iterates of holomorphic maps on convex domains in* $\mathbb{C}^n$, Trans. Amer. Math. Soc. (to appear).

[Mi1] K. Miller, *Barriers on cones for uniformly elliptic operators*, Ann. Mat. Pura Appl. **76** (1967), 93–105.

[Mi2] ——, *Extremal barriers on cones with Phragmen-Lindelöf theorems and other applications*, Ann. Mat. Pura Appl. **90** (1971), 297–329.

[O] J. K. Oddson, *On the boundary point principle for elliptic equations in the plane*, Bull. Amer. Math. Soc. **74** (1968), 666–670.

[P] S. I. Pinčuk, *On proper holomorphic mappings of strictly pseudoconvex domains*, Siberian Math. J. **15** (1974), 644–649.

[R1] R. M. Range, *On the topological extension to the boundary of biholomorphic maps in* $\mathbb{C}^n$, Trans. Amer. Math. Soc. **216** (1976), 203–216.

[R2] ——, *The Caratheodory metric and holomorphic maps on a class of weakly pseudoconvex domains*, Pacific J. Math. **78** (1978), 173–189.

[Roy] H. L. Royden, *Remarks on the Kobayashi metric*, Several Complex Variables. II (Maryland, 1970), Springer, Berlin, 1971, pp. 125–137.

[Ru] W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Grundlehren Math. Wiss., vol. 241, Springer-Verlag, Berlin, 1980.

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