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A CHARACTERIZATION OF HARMONIC ARAKELYAN SETS

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ABSTRACT. Given a closed subset F of either R^N, $N \geq 3$, or a Riemann surface, necessary and sufficient conditions are given so that every function continuous on F and harmonic in the interior can be uniformly approximated on F by globally defined harmonic functions.

1. INTRODUCTION

At a 1979 NATO conference in Durham, U.K., the following problem was posed by Goldstein [5]: Let F be a closed subset of Euclidean space R^N, $N \geq 2$. Call F a set of harmonic approximation if every function continuous on F and harmonic in the interior of F can be uniformly approximated there by harmonic functions on R^N. Give necessary and sufficient conditions so that F is a set of harmonic approximation. In this direction necessary and sufficient conditions in R^N have been given by Shaginyan [11], but only if F is nowhere dense. More recently, Bagby and Gauthier [3] have given other necessary and sufficient conditions on Riemann surfaces for arbitrary closed sets. Also, Shaginyan [12] has stated, without giving any proofs, results for tangential approximation on closed sets in R^N. Still more recently, Shaginyan and Ladouceur [13] have stated necessary and sufficient conditions for arbitrary closed sets in R^N, but their conditions are incorrect, as can be shown by an example.

In this paper we completely characterize sets F of harmonic approximation in R^N, $N \geq 3$, as well as on Riemann surfaces. Such sets F are also called harmonic Arakelyan sets since they are harmonic analogues of closed sets in the plane for which the Arakelyan Theorem [1] for uniform approximation by entire functions applies. Before discussing our results we state some preliminaries.

Let E be a nonempty subset of Ω, where Ω denotes either R^N, $N \geq 3$, or a Riemann surface. Let $E^* = E \cup \{*\}$ be the Alexandroff one-point compactification of E, where $*$ denotes the ideal point of E. We denote the interior, closure, and finite boundary of a set E in Ω by E^0, $\overline{E}$, and ∂E, respectively; while $\widehat{E}$ denotes the union of E and all of the relatively compact components of $\Omega - E$. The Euclidean norm of $x \in R^N$ is denoted $\|x\|$.

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Let $C(E)$ be the set of all real-valued continuous functions on E, and $H(E)$ the set of real-valued functions harmonic on some open set containing E. Functions in $H(\Omega)$ are called entire harmonic functions. If F is a closed subset of Ω we define $A(F) \equiv C(F) \cap H(F^0)$, while $\overline{H}_F(\Omega)$ (respectively, $\overline{H}(F)$) will denote the uniform limits on F of functions in $H(\Omega)$ (respectively, uniform limits of functions in $H(F)$).

The main goal of this paper is to prove

Theorem 1. *Let F be a closed subset of Ω, where either $\Omega = R^N$, $N \geq 3$, or Ω is a Riemann surface. Then $A(F) = \overline{H}_F(\Omega)$ if and only if the following conditions are satisfied*:

(1.1) *$\Omega - \widehat{F}$ and $\Omega - (\widehat{F})^0$ are thin at the same points.*

(1.2) *Every function $u \in A(F)$ can be extended to a function $\hat{u} \in A(\widehat{F})$.*

(1.3) *$\Omega^* - \widehat{F}$ is locally connected.*

A proof of Theorem 1 is given in §2.

Remark 1. In the case of a Riemann surface Theorem 1 reduces to a theorem of Bagby and Gauthier [3] because Theorem 1(1.1) is easily seen to be satisfied on Riemann surfaces, since thinness is a local property, and thus [9, Theorem 10.14] applies. Hence, we shall confine ourselves to proving Theorem 1 in the case $N \geq 3$.

It should be noted that, compared to the Bagby-Gauthier Theorem [3] mentioned above, our theorem requires a thinness condition (1.1) analogous to that of Deny [6] for uniform harmonic approximation in a neighborhood of a compact set. Furthermore, since there is a substantial difference in the notion of thinness between dimension 2 and higher dimensions, the proof of Theorem 1 is quite different from that of the Bagby-Gauthier theorem.

Remark 2. Theorem 1 is a harmonic analogue of the following.

Theorem (Arakelyan [1]). *Let F be a closed set in the complex plane $\mathbf{C}$. Then the necessary and sufficient conditions so that every function $f(z)$ continuous on F and analytic in F^0 is uniformly approximable on F by entire functions are*

(i) *$\mathbf{C}^* - F$ is connected,*

(ii) *$\mathbf{C}^* - F$ is locally connected.*

Remark 3. As noted earlier by Goldstein [5] Theorem 1 has applications to Problem 9.7 of Rubel [5] as well as other applications.

Remark 4. From Theorem 2 of Gauthier, Goldstein, and Ow [7] it follows that, for closed subsets F of Riemann surfaces Ω, the following is a necessary condition for $A(F) = \overline{H}_F(\Omega)$: For each bounded open set V such that $\partial V \subset F$, either $V \subset F$ or $V \cap F = \varnothing$.

This condition is also necessary in dimension $N \geq 3$, but since it is not needed in the proof of Theorem 1 we shall omit its proof.

2. Proof of Theorem 1

2a. **Sufficiency.** Assume that conditions (1.1), (1.2), and (1.3) hold. Let $u \in A(F)$ and a constant $\varepsilon > 0$ be given. By (1.2) we may extend u to a function $\hat{u} \in A(\widehat{F})$. By (1.1) it follows from [10, Theorem 3.10] that $A(\widehat{F}) = \overline{H}(\widehat{F})$.

Hence, there exists a $w \in H(\widehat{F})$ such that $|\hat{u} - w| < \varepsilon/2$ on $\widehat{F}$. Further, since $(R^N)^* - \widehat{F}$ is connected and by (1.3) locally connected, there exists by a theorem of Gauthier, Goldstein, and Ow [8] a $v \in H(R^N)$ such that $|w - v| < \varepsilon/2$ on $\widehat{F}$. Hence, $|\hat{u} - v| < \varepsilon$ on $\widehat{F}$ and this implies $|u - v| < \varepsilon$ on F. Thus, $A(F) = \overline{H}_F(R^N)$ as was to be shown.

2b. **Necessity.** Assume that $A(F) = \overline{H}_F(R^N)$. We first show that condition (1.1) is necessary. We first claim $A(\widehat{F}) = \overline{H}(\widehat{F})$. To see this, let $u \in A(\widehat{F})$. Then u is continuous on F. Further, if $p \in F^0$, then F and hence $\widehat{F}$ contain a neighborhood of p; and so u is harmonic at p and thus $u \in A(F)$. By assumption, u is the uniform limit on F of a sequence $\langle h_n \rangle$ of entire harmonic functions. It remains to show that u is the uniform limit of $\langle h_n \rangle$ on $\widehat{F}$. Let ε be a positive constant. Then there exists an integer J such that $|u - h_k| < \varepsilon$ on F whenever $k \geq J$. If E is any relatively compact component of $R^N - F$, then $\partial E \subset F$ implies $|u - h_k| < \varepsilon$ on ∂E and hence $|h_m - h_k| < 2\varepsilon$ on ∂E if $m, k \geq J$. By the maximum principle $|h_m - h_k| < 2\varepsilon$ holds on $\overline{E}$ and thus $|h_m - h_k| < 2\varepsilon$ on $\widehat{F}$. Hence, there exists a function h on $\widehat{F}$ such that $h_n \to h$ uniformly on $\widehat{F}$. So $h \in A(\widehat{F})$, and since $h \equiv u$ on the boundary of each relatively compact component of $R^N - F$, the maximum principle again implies $u \equiv h$ on $\widehat{F}$ and hence $A(\widehat{F}) = H(\widehat{F})$. Finally, by [10, Theorem 3.10] $A(\widehat{F}) = H(\widehat{F})$ implies condition (1.1), as was to be shown.

Before showing the necessity of condition (1.2) we first prove

Lemma 1. *If $A(F) = \overline{H}_F(R^N)$, $N \geq 2$, then necessarily $\partial F = \partial \widehat{F}$.*

Proof. For a proof of the case $N = 2$ see [7].

We now prove the necessity of $\partial F = \partial \widehat{F}$ when $N \geq 3$. Observe that $\partial \widehat{F} \subset \partial F$ and that $\partial F - \partial \widehat{F} = \partial F \cap (\widehat{F})^0$. To obtain a contradiction we suppose that there exists a point $p \in (\widehat{F})^0 \cap \partial F$. Then there exists a neighborhood $B_r(p) = \{q \mid \|p - q\| < r\} \subset (\widehat{F})^0$. Within $B_r(p)$ there are points $q \in R^N - F$ arbitrarily close to p since $p \in \partial F$. Fix a $q_0 \in (R^N - F) \cap B_r(p)$. Now the function $u(x) = \|x - q_0\|^{2-N} \in A(F)$. Let ε be a positive constant satisfying

$$0 < \varepsilon < 1. \tag{1}$$

By hypothesis there exists an $h \in H(R^N)$ such that

$$|h - u| < \varepsilon \quad \text{on } F. \tag{2}$$

Since u is nonconstant on F we may assume h is also nonconstant on F. We initially choose $q_0 \in (R^N - F) \cap B_r(p)$ so that

$$\|p - q_0\| < \tfrac{r}{2}. \tag{3}$$

Then $\|x - q_0\| \geq \frac{r}{2}$ for $x \in R^N - B_r(p)$ and so $u(x) = \|x - q_0\|^{2-N} \leq (\frac{r}{2})^{2-N}$ for $x \in R^N - B_r(p)$. Hence, by (1) and (2),

$$|h| \leq |h - u| + |u| \leq \varepsilon + (\tfrac{r}{2})^{2-N} < 1 + (\tfrac{r}{2})^{2-N}, \qquad x \in F - B_r(p). \tag{4}$$

On the other hand, on $F \cap B_r(p)$ we have by (1), (2) that

$$|h| \geq |u| - |h - u| \geq |u| - \varepsilon \geq |u| - 1. \tag{5}$$

In addition to q_0 satisfying (3) we may further choose q_0 so close to p that

$$\sup_{x\in F\cap B_r(p)} |u(x)| > 2[1+(\tfrac{r}{2})^{2-N}]+1. \tag{6}$$

By (5), (6),

$$\sup_{x\in F\cap B_r(p)} |h(x)| \geq 2[1+(\tfrac{r}{2})^{2-N}]. \tag{7}$$

Finally, (4), (7) imply

$$\sup_{x\in F} |h(x)| \text{ occurs in } F\cap B_r(p). \tag{8}$$

Assume first that F is compact, and so $\widehat{F}$ is compact. By the maximum principle $\sup_{\widehat{F}}|h| = \sup_F |h|$. But by (8), $\sup_F |h|$ occurs at a point of $F\cap B_r(p)$ all of which are interior to $\widehat{F}$. Hence, $\sup_{\widehat{F}}|h|$ occurs at an interior point of $\widehat{F}$, contradicting h being nonconstant on F.

Next assume F is unbounded. Choose an exhaustion of R^N by balls T_j, $\overline{T}_j \subset T_{j+1}$, $R^N = \bigcup_1^\infty T_j$ so $\partial T_j \cap F \neq \varnothing$ for all j. Further, choose j large enough so that $\overline{B_r(p)} \subset T_j$ and $u(x) = \|x-q_0\|^{2-N} < 1$ for all $x \in R^N - T_j$. We apply the earlier compact case to $F\cap \overline{T}_j$ and again obtain a contradiction by violating the maximum principle. This completes the proof of the lemma.

We now show the necessity of condition (1.2). Assume $A(F) = \overline{H}_F(R^N)$ and let $u \in A(F)$. Now $\partial F = \partial \widehat{F}$ by Lemma 1 and hence $u|_{\partial F} = u|_{\partial \widehat{F}}$. Since $A(F) = \overline{H}_F(R^N)$, we have by the maximum principle that $u|_{\partial\widehat{F}}$ extends to a function $\hat{u} \in A(\widehat{F})$ and this $\hat{u}$ is the desired extension of u to $\widehat{F}$. Note by Lemma 1 that no points of $\partial F \cap (\widehat{F})^0$ are involved in the extension since $\partial F \cap (\widehat{F})^0 = \partial F - \partial \widehat{F} = \varnothing$.

Finally, we show that condition (1.3) is necessary, We first need

Lemma 2 (Bagby-Gauthier [4]). *Let W be a bounded domain in R^N, $N \geq 2$, and $q \in W$. Denote by $g_W(\cdot, q)$ the Green function for W with singularity at q. Let $u : \overline{W} - \{q\} \to \mathbb{R}$ be continuous, harmonic on $W - \{q\}$, and suppose $u - g_W(\cdot, q)$ has a removable singularity at q. Then $u(p) \geq 0$ for all $p \in \partial W$ implies $u(p) - g_W(p, q) \geq 0$ for all $p \in W$.*

We now show that (1.3) is necessary. Assume $A(F) = \overline{H}_F(R^N)$ and, to obtain a contradiction, suppose $(R^N)^* - \widehat{F}$ is not locally connected. Then there exists a neighborhood $U = \{p \mid \|p\| > R > 1\}$ of $*$ and there exists points $p \in (R^N)^* - \widehat{F}$, $p \neq *$, with $\|p\|$ arbitrarily large and which cannot be joined to $*$ by a curve in $(R^N)^* - \widehat{F}$ without hitting the ball $B_R(0) = \{p \mid \|p\| < R\}$. Each of these points p thus belongs to a bounded component of $[(R^N)^* - \widehat{F}] \cap B_R(0)$ whose boundary consists of points of $\widehat{F}$ and the sphere $S_R(0) = \{p \mid \|p\| = R\}$. We choose such a sequence $\langle b_m \rangle$ such that

$$m \leq \|b_m\|, \qquad \|b_m\| < \tfrac{1}{3}\|b_{m+1}\|, \tag{9}$$

and each b_m belongs to a bounded component D_m of $[(R^N)^* - \widehat{F}] \cap B_R(0)$, with $2\|b_m\| > \sup_{p\in D_m}\|p\|$. The conditions (9) imply that the components D_m are pairwise disjoint. Also note $S_R(0) \cap \partial D_m \neq \varnothing$ for each m since

otherwise $\partial D_m \subset \widehat{F}$, a contradiction, so we may choose a sequence $\langle a_m \rangle$ such that $a_m \in D_m$, $a_m \neq b_m$, and $\operatorname{dist}(a_m, S_R(0)) < \frac{1}{m}$. We may assume $\langle a_m \rangle$ converges to some point $a \in S_R(0)$.

We follow the method of proof of Theorem 3.3 in [4] and let $g_m \equiv g_{D_m}(\cdot, b_m)$, $m \geq 1$, be the Green function for D_m with singularity at b_m. For each $m \geq 1$ choose a finite constant c_m such that $c_m > m/g_m(a_m)$. By Lemma 2.6 in [2] there exists a function u harmonic in $R^N - \bigcup_{m=1}^{\infty} b_m$ such that for each m the harmonic function $u - c_m g_m$ has a removable singularity at b_m. Note that $A(F) = \overline{H}_F(R^N)$ implies $A(\widehat{F}) = \overline{H}_{\widehat{F}}(R^N)$ (cf. [4, Remark 4.1b]). Now $u \in A(\widehat{F})$ and so by assumption there exists a function v harmonic in R^N such that $0 \leq u - v \leq 1$ on $\widehat{F}$. Let $c = \min_{S_{R(0)}}(u - v)$ and $d = \min(0, c)$. Recall $\partial D_m \subset \widehat{F} \cup S_R(0)$. Note that $u - v - d \geq 0$ throughout ∂D_m. Since $u - v - d$ is continuous on $\overline{D}_m - \{b_m\}$ and harmonic on $D_m - \{b_m\}$, and $u - v - d - c_m g_m$ has a removable singularity at b_m, we conclude by Lemma 2 that $u - v - d - c_m g_m \geq 0$ throughout D_m. In particular, at a_m we have $u(a_m) - v(a_m) \geq c_m g_m(a_m) + d \geq m + d$. Hence $u(a) - v(a) = \lim_{m \to \infty} u(a_m) - v(a_m) = \infty$, which is a contradiction. This completes the proof of the necessity in Theorem 1.

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