

ON THE SOLUTION OF THE DIRICHLET PROBLEM FOR THE TWO-DIMENSIONAL LAPLACE EQUATION

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ABSTRACT. The solution of the Dirichlet problem for the two-dimensional Laplace equation is obtained as a modified single layer potential by a method applicable even when the logarithmic capacity of the boundary curve is equal to 1.

1. INTRODUCTION

A classical method for solving the Dirichlet problem for the Laplace equation consists in seeking the solution in the form of a double layer potential and reducing the problem to a Fredholm integral equation of the second kind on the boundary of the domain. In [1] it was shown that the boundary integral operators associated with the harmonic potentials have certain composition properties that enable us to seek the solution of the Dirichlet problem also as a single layer potential. This leads to a Fredholm integral equation of the first kind. The method is successful in \mathbf{R}^3 (the case discussed in [1]), but breaks down in \mathbf{R}^2 , because when the logarithmic capacity [2] of the boundary curve is equal to 1, the corresponding homogeneous equation has nonzero solutions.

Below we propose a simple remedy that appears to eliminate this drawback, without interfering with the form of the solution as a single layer potential.

The advantage of a technique of this type over the classical one is that here both the interior and exterior problems are solved by means of one and the same set of boundary integral equations. For example, in Fichera's method [3] (see also [4] for a good discussion and illustration) there are two such equations, involving the unknown density function and an unknown additive constant. The method proposed in this paper seems to have the further advantage that it reduces the problem to a single integral equation, and does not require the computation of any unknown constant that needs to be added to the harmonic potential. Also, the existence and uniqueness proof avoids differentiation along the boundary curve (used by Fichera) and can be extended to the Dirichlet problem for certain two-dimensional elliptic systems with constant coefficients.

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2. HARMONIC POTENTIALS

Let S^+ be a domain in \mathbf{R}^2 bounded by a closed C^2 -curve ∂S , and let $S^- = \mathbf{R}^2 \setminus (S^+ \cup \partial S)$. We define the harmonic single and double layer potentials by

$$(1) \quad \begin{aligned} (v\varphi)(x) &= -\frac{1}{2\pi} \int_{\partial S} (\ln|x-y|)\varphi(y) ds(y), \\ (w\psi)(x) &= -\frac{1}{2\pi} \int_{\partial S} \left(\frac{\partial}{\partial \nu(y)} \ln|x-y| \right) \psi(y) ds(y), \end{aligned}$$

where $x = (x_1, x_2)$ is a generic point in \mathbf{R}^2 , $|x-y| = [(x_1-y_1)^2 + (x_2-y_2)^2]^{1/2}$, ν is the unit outward normal to ∂S , and the densities φ and ψ are such that the above integrals exist even for $x \in \partial S$. When $x \in \partial S$, we denote these potentials by $v_0\varphi$ and $w_0\psi$.

Theorem 1. (i) If $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1)$, then the functions

$$(2) \quad \mu^+(\varphi) = (v\varphi)|_{S^+}, \quad \mu^-(\varphi) = (v\varphi)|_{S^-}$$

belong to $C^{1,\alpha}(\bar{S}^+)$ and $C^{1,\alpha}(\bar{S}^-)$, respectively, and

$$(3) \quad \frac{\partial \mu^+(\varphi)}{\partial \nu} = \frac{1}{2}\varphi + w_0^*\varphi, \quad \frac{\partial \mu^-(\varphi)}{\partial \nu} = -\frac{1}{2}\varphi + w_0^*\varphi \quad (\text{on } \partial S),$$

where w_0^* is the adjoint of w_0 , defined by

$$(w_0^*\varphi)(x) = -\frac{1}{2\pi} \int_{\partial S} \left(\frac{\partial}{\partial \nu(x)} \ln|x-y| \right) \varphi(y) ds(y), \quad x \in \partial S.$$

(ii) If $\psi \in C^{1,\alpha}(\partial S)$, $\alpha \in (0, 1)$, then the functions

$$(4) \quad \theta^+(\psi) = \begin{cases} (w\psi)|_{S^+} & \text{in } S^+, \\ -\frac{1}{2}\psi + w_0\psi & \text{on } \partial S, \end{cases} \quad \theta^-(\psi) = \begin{cases} (w\psi)|_{S^-} & \text{in } S^-, \\ \frac{1}{2}\psi + w_0\psi & \text{on } \partial S \end{cases}$$

belong to $C^{1,\alpha}(\bar{S}^+)$ and $C^{1,\alpha}(\bar{S}^-)$, respectively.

(In both (i) and (ii) the derivatives on ∂S are 'one-sided'.)

For the proof of this assertion see, for example, Remarks 1.49 and 1.50 and Theorem 1.51 in [5, p. 41] (where v and w are defined without the factor $(2\pi)^{-1}$).

If instead of $-\frac{1}{2\pi} \ln|x-y|$ we take the fundamental solution

$$-\frac{1}{2\pi} (\ln|x-y| + c), \quad c = \text{const},$$

then we can define the modified single layer potential

$$(v^c\varphi)(x) = -\frac{1}{2\pi} \int_{\partial S} (\ln|x-y| + c)\varphi(y) ds(y) = (v\varphi)(x) + c p\varphi,$$

where

$$p\varphi = -\frac{1}{2\pi} \int_{\partial S} \varphi ds.$$

Clearly, the corresponding modified double layer potential $w^c\psi$ coincides with $w\psi$.

It is well known [2] that there is a unique density $\tilde{\varphi}$ such that

$$(5) \quad (v_0\tilde{\varphi})(x) = \omega = \text{const}, \quad p\tilde{\varphi} = -\frac{1}{2\pi}.$$

The numbers $2\pi\omega$ and $e^{-2\pi\omega}$ are called Robin's constant and the logarithmic capacity of ∂S , respectively.

Since, as $|x| \rightarrow \infty$,

$$\ln|x - y| = \ln|x| + O(|x|^{-1}),$$

from (1)₁ and (5)₂ it follows that

$$(6) \quad \begin{aligned} (v\varphi)(x) &= (p\varphi) \ln|x| + O(|x|^{-1}), \\ (v\tilde{\varphi})(x) &= -\frac{1}{2\pi} \ln|x| + O(|x|^{-1}) \quad \text{as } |x| \rightarrow \infty. \end{aligned}$$

Theorem 2. *If $\varphi \in C^{0,\alpha}(\partial S)$ and $c \neq 2\pi\omega$, then $v_0^c\varphi = 0$ implies that $\varphi = 0$.*

Proof. Let u be the operator defined on $C^{0,\alpha}(\partial S)$ by

$$(7) \quad u\varphi = v^c\varphi + 2\pi(p\varphi)v\tilde{\varphi} - 2\pi\omega p\varphi.$$

From (5)–(7) and the assumption that $v_0^c\varphi = 0$ we see that

$$\begin{aligned} \Delta(u\varphi) &= 0 \quad \text{in } S^-, \\ u\varphi &= 0 \quad \text{on } \partial S, \\ (u\varphi)(x) &= (c - 2\pi\omega)p\varphi + O(|x|^{-1}) \quad \text{as } |x| \rightarrow \infty. \end{aligned}$$

The uniqueness of the solution of the exterior Dirichlet problem clearly requires that $(c - 2\pi\omega)p\varphi = 0$, and, since $c \neq 2\pi\omega$, we deduce that $p\varphi = 0$.

Thus, we have $v^c\varphi = v\varphi$, and, by (6)₁,

$$\begin{aligned} \Delta(v\varphi) &= 0 \quad \text{in } S^+ \cup S^-, \\ v_0\varphi &= 0 \quad (\text{on } \partial S), \\ v\varphi &= O(|x|^{-1}) \quad \text{as } |x| \rightarrow \infty. \end{aligned}$$

Using uniqueness arguments again, this time for both the interior and the exterior Dirichlet problems, we find that $v\varphi = 0$ in \mathbf{R}^2 . By (2), this means that $\partial\mu^+(\varphi)/\partial\nu = \partial\mu^-(\varphi)/\partial\nu = 0$. Subtracting (3)₂ from (3)₁, we now conclude that $\varphi = 0$.

3. BOUNDARY INTEGRAL OPERATORS

The properties of the boundary operators discussed in [1] can also be established in \mathbf{R}^2 for the modified potentials.

Let φ be arbitrary in $C^{0,\alpha}(\partial S)$. By Theorem 1, we can write

$$(8) \quad \mu^+(\varphi)|_{\partial S} = f \in C^{1,\alpha}(\partial S), \quad \left. \frac{\partial\mu^+(\varphi)}{\partial\nu} \right|_{\partial S} = g \in C^{0,\alpha}(\partial S).$$

Since $\mu^+(\varphi)$ is harmonic in S^+ , it admits the Green representation

$$(9) \quad \begin{aligned} \mu^+(\varphi) &= v g - w f \quad \text{in } S^+, \\ \frac{1}{2} f &= v_0 g - w_0 f \quad (\text{on } \partial S). \end{aligned}$$

In view of (2) and (4), this can also be written as

$$\mu^+(\varphi) = \mu^+(g) - \theta^+(f) \quad \text{on } \bar{S}^+.$$

By (8) and Theorem 1, all the terms in this relation belong to $C^{1,\alpha}(\bar{S}^+)$. Taking their normal derivatives on ∂S , replacing $\partial\mu^+(\varphi)/\partial\nu$ from (8)₂ and $\partial\mu^+(g)/\partial\nu$ from (3)₁, and using the notation

$$(10) \quad n_0 f = \frac{\partial\theta^+(f)}{\partial\nu} \in C^{0,\alpha}(\partial S),$$

we obtain

$$(11) \quad \frac{1}{2}g = w_0^* g - n_0 f.$$

Also, combining (3)₁ and (8)₂, we can write

$$(12) \quad g = \frac{1}{2}\varphi + w_0^* \varphi.$$

Replacing $f = \mu^+(\varphi)|_{\partial S} = v_0 \varphi$ and g from (12) in (9)₂ and (11), and taking the arbitrariness of φ into account, we now find that

$$(13) \quad w_0 v_0 = v_0 w_0^*, \quad n_0 v_0 = w_0^{*2} - \frac{1}{4}I \quad \text{on } C^{0,\alpha}(\partial S),$$

where I is the identity operator.

Starting from the fact that $\theta^+(\psi)$, $\psi \in C^{1,\alpha}(\partial S)$, is harmonic in S^+ and proceeding in exactly the same way as above, we obtain

$$(14) \quad n_0 w_0 = w_0^* n_0, \quad v_0 n_0 = w_0^2 - \frac{1}{4}I \quad \text{on } C^{1,\alpha}(\partial S).$$

Theorem 3. *The relations (13) and (14) remain valid if v_0 is replaced by v_0^c .*

Proof. Substituting $v_0 = v_0^c - cp$ in (13) and (14), we see that the assertion is proved if we show that

$$\begin{aligned} w_0 p &= p w_0^*, \quad n_0 p = 0 \quad \text{on } C^{0,\alpha}(\partial S), \\ p n_0 &= 0 \quad \text{on } C^{1,\alpha}(\partial S). \end{aligned}$$

(By Theorem 1, all these compositions are well defined.)

The double layer potential satisfies (see formulae (1.39) and (1.40) in [5, p. 37])

$$(15) \quad w_1 = -1 \quad \text{in } S^+, \quad w_0 1 = -\frac{1}{2} \quad (\text{on } \partial S),$$

in other words,

$$\theta^+(1) = -1 \quad \text{on } \bar{S}^+.$$

Since $p\varphi$ is a constant, from (1)₂, (10), (4)₁, (15), and the definitions of p and w_0^* it follows that for $\varphi \in C^{0,\alpha}(\partial S)$

$$\begin{aligned} (p w_0^*) \varphi &= p(w_0^* \varphi) = \int_{\partial S} \left[-\frac{1}{2\pi} \int_{\partial S} \left(\frac{\partial}{\partial\nu(x)} \ln|x-y| \right) \varphi(y) ds(y) \right] ds(x) \\ &= \int_{\partial S} \left[-\frac{1}{2\pi} \int_{\partial S} \frac{\partial}{\partial\nu(x)} \ln|x-y| ds(x) \right] \varphi(y) ds(y) \\ &= \int_{\partial S} (w_0 1) \varphi ds = -\frac{1}{2} p \varphi = (p \varphi) w_0 1 = w_0 (p \varphi) = (w_0 p) \varphi, \\ (n_0 p) \varphi &= n_0 (p \varphi) = (p \varphi) n_0 1 = (p \varphi) \frac{\partial\theta^+(1)}{\partial\nu} = (p \varphi) \frac{\partial}{\partial\nu} (-1) = 0, \end{aligned}$$

and for $\psi \in C^{1,\alpha}(\partial S)$

$$(pn_0)\psi = p(n_0\psi) = p\left(\frac{\partial\theta^+(\psi)}{\partial\nu}\right) = \int_{\partial S} \frac{\partial\theta^+(\psi)}{\partial\nu} ds = \int_{S^+} \Delta(w\psi) d\sigma = 0,$$

as required.

Other useful properties of the boundary integral operators are gathered in the following assertion.

Theorem 4. (i) $\frac{1}{4}$ is an eigenvalue of w_0^2 , and $\varphi = \text{const}$ the only corresponding eigenfunctions.

(ii) $\frac{1}{4}$ is an eigenvalue of w_0^{*2} , and the corresponding eigenspace coincides with that of w_0^* for the eigenvalue $-\frac{1}{2}$.

(iii) If $n_0\psi = 0$, then $\psi = \text{const}$.

Proof. (i) The result follows from the fact that

$$(w_0^2 - \frac{1}{4}I)\varphi = (w_0 - \frac{1}{2}I)((w_0 + \frac{1}{2}I)\varphi) = 0$$

implies that $(w_0 + \frac{1}{2}I)\varphi = 0$ (since $\frac{1}{2}$ is not an eigenvalue of w_0), which, in turn, implies that $\varphi = \text{const}$ (since $-\frac{1}{2}$ is an eigenvalue of w_0 , the corresponding eigenfunctions being arbitrary constants).

(ii) This is proved in exactly the same way as (i).

(iii) By $(14)_2$,

$$0 = v_0(n_0\psi) = (v_0n_0)\psi = (w_0^2 - \frac{1}{4}I)\psi,$$

and (i) yields $\psi = \text{const}$.

4. INTERIOR DIRICHLET PROBLEM

We seek the solution of the boundary value problem

$$(16) \quad \begin{aligned} \Delta u &= 0 && \text{in } S^+, \\ u &= f && \text{on } \partial S \end{aligned}$$

in the form of a modified single layer potential $v^c\varphi$ with $c \neq 2\pi\omega$ chosen a priori. Then the problem reduces to the solution of the Fredholm equation of the first kind

$$(17) \quad v_0^c\varphi = f.$$

Theorem 5. The integral equation (17) has a unique $C^{0,\alpha}$ -solution φ if and only if $f \in C^{1,\alpha}(\partial S)$. In this case, φ also satisfies

$$(18) \quad (w_0^{*2} - \frac{1}{4}I)\varphi = n_0f.$$

Proof. If φ is a $C^{0,\alpha}$ -solution of (17), then, by Theorem 1, $f \in C^{1,\alpha}(\partial S)$. Consequently, we can apply n_0 to both sides of (17) and use Theorem 3 and $(13)_2$ to obtain (18).

Conversely, suppose that $f \in C^{1,\alpha}(\partial S)$. Then, by (10) and $(4)_1$,

$$\int_{\partial S} n_0f ds = \int_{\partial S} \frac{\partial\theta^+(f)}{\partial\nu} ds = \int_{S^+} \Delta(\theta^+(f))d\sigma = \int_{S^+} \Delta(wf) d\sigma = 0.$$

Hence, by Theorem 4(i), (ii) and Fredholm's Alternative, equation (18) is solvable in $C^{0,\alpha}(\partial S)$ and its general solution is $\varphi = \varphi_0 + a\tilde{\varphi}$, where φ_0 is a

particular (fixed) solution of (18) and a an arbitrary constant. Using Theorem 3 and (13)₂, we can now write

$$0 = (w_0^{*2} - \frac{1}{4}I)(\varphi_0 + a\tilde{\varphi}) - n_0f = n_0[v_0^c(\varphi_0 + a\tilde{\varphi}) - f].$$

By Theorem 4(iii), this implies that

$$v_0^c(\varphi_0 + a\tilde{\varphi}) - f = a' = \text{const},$$

which, in view of (5), can be rewritten in the form

$$a' = v_0^c\varphi_0 + a(v_0\tilde{\varphi} + cp\tilde{\varphi}) - f = v_0^c\varphi_0 + a\left(\omega - \frac{1}{2\pi}c\right) - f.$$

Hence,

$$v_0^c\varphi_0 - f = a' - a\left(\omega - \frac{1}{2\pi}c\right) = a'',$$

where a'' is a specific constant. Setting $a''' = 2\pi a'' / (c - 2\pi\omega)$, we now conclude that $\varphi = \varphi_0 + a'''\tilde{\varphi} \in C^{0,\alpha}(\partial S)$ is a solution of (17).

Let φ_1 and φ_2 be two solutions of (17). Then $\varphi = \varphi_1 - \varphi_2$ satisfies $v_0^c\varphi = 0$; therefore, by Theorem 2, $\varphi = 0$, which implies that the solution of (17) is unique.

5. COMMENT ON UNIQUENESS

Clearly, the representation of the solution of (16) as $u = v^c\varphi$ is not unique, since the density φ depends on the choice of the constant c . However, u itself is unique, so, for any two distinct numbers c_1 and c_2 , $c_1 \neq 2\pi\omega$, $c_2 \neq 2\pi\omega$, the corresponding densities φ_1 and φ_2 must satisfy $v_0^{c_1}\varphi_1 = v_0^{c_2}\varphi_2 = f$. By the definition of $v^c\varphi$, this means that

$$v_0(\varphi_1 - \varphi_2) = c_2p\varphi_2 - c_1p\varphi_1 = \text{const},$$

hence, in view of (5)₁,

$$(19) \quad \varphi_1 - \varphi_2 = k\tilde{\varphi}, \quad c_2p\varphi_2 - c_1p\varphi_1 = k\omega, \quad k = \text{const}.$$

Then, by (5)₂,

$$(20) \quad p\varphi_1 - p\varphi_2 = -\frac{1}{2\pi}k.$$

From (19)₂ and (20) it follows that $2\pi\omega(p\varphi_1 - p\varphi_2) = c_1p\varphi_1 - c_2p\varphi_2$, which yields

$$p\varphi_2 = \frac{c_1 - 2\pi\omega}{c_2 - 2\pi\omega}p\varphi_1.$$

Using (20) again, we now find that

$$k = 2\pi \frac{c_1 - c_2}{c_2 - 2\pi\omega} p\varphi_1;$$

consequently, by (19)₁,

$$(21) \quad \varphi_2 = \varphi_1 + 2\pi \frac{c_2 - c_1}{c_2 - 2\pi\omega} (p\varphi_1)\tilde{\varphi}.$$

This equality shows how any solution $\varphi = \varphi(c)$ of the integral equation (17) can be generated once the solution for a particular value of $c \neq 2\pi\omega$ has been found.

6. EXTERIOR DIRICHLET PROBLEM

Consider the boundary value problem

$$(22) \quad \begin{aligned} \Delta u &= 0 \quad \text{in } S^-, \\ u &= f \quad \text{on } \partial S, \\ u(x) &= q \ln|x| + O(1) \quad \text{as } |x| \rightarrow \infty, \quad q = \text{const.} \end{aligned}$$

We know already that there is a unique function $u = v^c \varphi$, $c \neq 2\pi\omega$, satisfying the first two relations (22). However, (22)₃ might not hold for this function, so instead we take $u = v^{c'} \varphi'$, where $\varphi' = \varphi + \alpha \tilde{\varphi}$, $\alpha = \text{const}$. Then, by (5),

$$\begin{aligned} \Delta(v^{c'} \varphi') &= 0 \quad \text{in } S^-, \\ v_0^{c'} \varphi' &= v_0 \varphi + c' p \varphi + \alpha(v \tilde{\varphi} + c' p \tilde{\varphi}) \\ &= v_0^c \varphi + (c' - c)p \varphi + \alpha \left(\omega - \frac{1}{2\pi} c' \right) = f \end{aligned}$$

if

$$(23) \quad c' = 2\pi \frac{cp\varphi - \alpha\omega}{2\pi p\varphi - \alpha},$$

and

$$v^{c'} \varphi' = v\varphi + \alpha v \tilde{\varphi} + O(1) = \left(p\varphi - \frac{1}{2\pi} \alpha \right) \ln|x| + O(1) = q \ln|x| + O(1)$$

if

$$(24) \quad \alpha = 2\pi(p\varphi - q).$$

From (23) and (24) we obtain

$$(25) \quad c' = 2\pi\omega + \frac{1}{q}(c - 2\pi\omega)p\varphi.$$

Consequently, if $q \neq 0$, then the (unique) solution of (22) is

$$u = v^{c'} (\varphi + 2\pi(p\varphi - q)) \tilde{\varphi},$$

with c' given by (25).

When $q = 0$, the solution of (22) is unobtainable as above, since $\alpha = 2\pi p\varphi$ and

$$v_0^{c'} \varphi' = f - (c - 2\pi\omega)p\varphi$$

for any c' ; therefore, we cannot choose $c \neq 2\pi\omega$ so that $p\varphi = 0$: (21), which gives the arbitrariness in φ , yields

$$p\varphi_2 = \frac{c_1 - 2\pi\omega}{c_2 - 2\pi\omega} p\varphi_1.$$

In this case the solution is, clearly,

$$u = v^c (\varphi + 2\pi(p\varphi) \tilde{\varphi}) + (c - 2\pi\omega)p\varphi.$$

7. CONCLUSION

The above method enables us to solve both the interior and exterior problems by means of the same Fredholm equation of the first kind on ∂S , whose solution depends only on the boundary data and is expressed as a single layer potential (except if $q = 0$ in (22), when a (known) additive constant is necessary). In the interior problem, the arbitrary choice of $c \neq 2\pi\omega$ allows an additional condition, if needed, to be imposed on the density (for example, a special normalization). In the case of the exterior problem, the kernel and density of the potential are easily adjusted for the solution to comply with any required logarithmic asymptotic behaviour. This is particularly useful, for example, if the physical model we study needs the problem to be solved repeatedly, with the same data on ∂S but different far-field constants in the logarithmic term.

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