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INVARIANT THEORY OF THE DUAL PAIRS $(\mathrm{SO}^*(2n), \mathrm{Sp}(2k, \mathbf{C}))$ AND $(\mathrm{Sp}(2n, \Re), \mathrm{O}(N))$

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ABSTRACT. Let $G \equiv \mathrm{Sp}(2k, \mathbf{C})$ or $\mathrm{O}(N)$ and $G' \equiv \mathrm{SO}^*(2n)$ or $\mathrm{Sp}(2n, \Re)$. The adjoint representation of G' on its Lie algebra $\mathscr{G}'$ gives rise to the coadjoint representation of G' on the symmetric algebra of all polynomial functions on $\mathscr{G}'$. The polynomials that are fixed by the restriction of the coadjoint representation to a block diagonal subgroup K' of G' form a subalgebra called the algebra of K'-invariants. Using the theory of invariants of Procesi for the "dual pair" (G', G), a finite set of generators of this algebra is explicitly determined.

1. INTRODUCTION

The theory of invariants which was first investigated by Cayley in *Memoire sur les Hyperdeterminants* (1846) and *Memoirs on quantics* (1854–1859) gained intensive interest when it was discovered that it was intimately related to the theory of group representations. It is not a coincidence that Weyl's celebrated *Classical groups* [We] is entitled *The classical groups, their invariants and representations.* Many treatises on invariant theory were written in recent periods (see [Fo, DC, Sp]), but few stress its importance in physics. In physics, if G is a symmetry group of some physical system, then the universal enveloping algebra $\mathscr{U}$ of its Lie algebra $\mathscr{G}$ is an algebra of tensor operators and the invariant operators which form the centralizer of $\mathscr{G}$ in $\mathscr{U}$ are called "Casimir invariants" (cf. [AM, BR]); this term was coined by the physicist Racah for his generalization of the quadratic invariant for semisimple Lie groups discovered by Casimir and Van der Waerden. The importance of these Casimir invariants is due to a fundamental theorem of Chevalley which states: "For every semisimple Lie algebra $\mathscr{G}$ of rank n, there exists a set of n invariants of generators whose eigenvalues characterize the finite-dimensional irreducible representations of $\mathscr{G}$", which in physical terms means that the spectra of the invariant operators associated with G determine the observable quantum numbers of the physical system. It can be shown that the ring of Casimir invariants is isomorphic to the ring of polynomials invariant under the adjoint representation of G via the canonical bijection (see [Di, Go]), and hence the determination of the Casimir invariants is equivalent to that of the polynomial invariants. In

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1963, Moshinsky discovered a further generalization of the Casimir operators which he used to resolve the multiplicity problem for two-fold tensor products of the unitary group $\mathrm{U}(3)$ [Mo]. Moshinsky and Quesne also discovered the notion of complementary pairs for several pairs of reductive groups [MQ]. The systematic studies of these pairs of groups led to the classification of the reductive dual pairs by Howe [Ho1]. The generalization of the idea of Moshinsky of using the generalized Casimir operators and the theory of dual pairs to resolve the multiplicity of arbitrary n-fold tensor products of $\mathrm{U}(N)$ is carried out in [KT3, KT4]. In [KT2], the invariant theory of invariants and generalized Casimir operators for the dual pair $\mathrm{GL}(p_1, \mathbf{C}) \times \cdots \times \mathrm{GL}(p_m, \mathbf{C})$ in $\mathrm{GL}(n, \mathbf{C})$, $p_1 + \cdots + p_m = n$, and $\mathrm{GL}(N, \mathbf{C}) \times \cdots \times \mathrm{GL}(N, \mathbf{C}) \supset \mathrm{GL}(N, \mathbf{C})$ was determined. In this paper, we shall give an explicit determination of a finite set of generators for the polynomial invariants and generalized Casimir operators for the dual pairs $(\mathrm{SO}^*(2n), \mathrm{Sp}(2k, \mathbf{C}))$ and $(\mathrm{Sp}(2n, \Re), \mathrm{O}(N))$. The eigenvalues of these generalized Casimir operators are then used to resolve the multiplicity problem for arbitrary n-fold tensor product representations of $\mathrm{Sp}(2k, \mathbf{C})$ in [Le] and of $\mathrm{SO}(N)$ in a forthcoming publication.

Let G be a linear Lie group and $\mathscr{G}$ be its Lie algebra. G acts on $\mathscr{G}$ via the adjoint representation, that is, $(g, X) \to g \cdot X = gXg^{-1}$, where $g \in G$, $X \in \mathscr{G}$.

Let $\mathbf{S}(\mathscr{G}^*)$ denote the symmetric algebra of all complex-valued polynomial functions on $\mathscr{G}$. Then we have the coadjoint representation of G on $\mathbf{S}(\mathscr{G}^*)$, namely,

$$(g, f) \to g \cdot f \qquad (g \in G, f \in \mathbf{S}(\mathscr{G}^*))$$

where $(g \cdot f)(X) = f(g^{-1} \cdot X) = f(g^{-1}Xg)$, $X \in \mathscr{G}$. Let K be a subgroup of G. Then a polynomial $f \in \mathbf{S}(\mathscr{G}^*)$ is said to be K-invariant if $k \cdot f = f$ for all $k \in K$.

Now, let $\mathrm{GL}(N, \mathbf{C})$ be the $N \times N$ complex general linear group. Consider the action $(g, (X_1, \dots, X_m)) \to g \cdot (X_1, \dots, X_m) \equiv (gX_1g^{-1}, \dots, gX_mg^{-1})$, where $X_i \in \mathrm{gl}(N, \mathbf{C})$, the Lie algebra of $\mathrm{GL}(N, \mathbf{C})$, and $g \in \mathrm{GL}(N, \mathbf{C})$. Let $\mathrm{Tr}(X_{i_1}, \dots, X_{i_k})$ be the trace of the product $X_{i_1}, \dots, X_{i_k}$, where $X_{i_j} \in \mathrm{gl}^{(i)}(N, \mathbf{C})$. The indices $i_1, \dots, i_k$ may not be distinct. Procesi has proved the following:

Theorem 1 (cf. [Pr]). *The algebra of polynomial invariants under the action of* $\mathrm{GL}(N, \mathbf{C})$ *on* $\mathrm{gl}^{(1)}(N, \mathbf{C}) \oplus \cdots \oplus \mathrm{gl}^{(m)}(N, \mathbf{C})$, *where* $\mathrm{gl}^{(i)}(N, \mathbf{C}) = \mathrm{gl}(N, \mathbf{C})$ *for* $i = 1, \dots, m$, *is finitely generated by the functions of the type* $\mathrm{Tr}(X_{i_1} \cdots X_{i_k})$, *where one may restrict to the case* $k \leq 2^N - 1$.

$(\mathrm{GL}(n, \mathbf{C}), \mathrm{GL}(N, \mathbf{C}))$ forms a dual pair of reductive groups, in the sense of [Ho1]. We have a "dual" theorem of the above theorem.

Let $(p_1, \dots, p_m)$ be an m-tuple of positive integers such that $p_1 + \cdots + p_m = n$, and consider the block diagonal subgroup $\mathrm{GL}(p_1, \mathbf{C}) \times \cdots \times \mathrm{GL}(p_m, \mathbf{C})$ of $\mathrm{GL}(n, \mathbf{C})$. Again, we have the adjoint representation of $\mathrm{GL}(n, \mathbf{C})$ on $\mathbf{S}(\mathrm{gl}(n, \mathbf{C})^*)$.

Let $Y \in \mathrm{gl}(n, \mathbf{C})$, and partition Y in block matrices of the form $[[Y]_{uv}]$, where $[Y]_{uv}$ is a $p_u \times p_v$ matrix, $1 \leq u, v \leq m$. In [KT1], Klink and Ton-That have established the following:

Theorem 2. *The algebra of all* $(\mathrm{GL}(p_1, \mathbf{C}) \times \cdots \times \mathrm{GL}(p_m, \mathbf{C}))$*-invariant polynomials is finitely generated by the constants and the functions of the form*:

$$\mathrm{Tr}([Y]_{u_1u_2}\,[Y]_{u_2u_3}\,[Y]_{u_i, u_{i+1}}\,\cdots\,[Y]_{u_qu_1}) ,$$

for all $u_i = 1, \ldots, m$, $i = 1, \ldots, q$, $u_{q+1} = u_1$.

Procesi has also established the polynomial invariants under the action of $\mathrm{Sp}(N, \mathbf{C})$ (in this case, $N = 2k$) and $\mathrm{O}(N)$ (N arbitrary) on $\mathrm{gl}^{(1)}(N, \mathbf{C}) \oplus \cdots \oplus \mathrm{gl}^{(m)}(N, \mathbf{C})$ as follows:

Theorem 3 (cf. [Pr]). *The polynomial invariants of the above action of G is finitely generated by the functions of type* $\mathrm{Tr}(A_{i_1} \cdots A_{i_\nu})$, $A_{i_j} = X_{i_j}$ *or* $X^*_{i_j}$ *where*

$$X^* = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix} X^{\mathrm{T}} \begin{bmatrix} 0 & -I_k \\ I_k & 0 \end{bmatrix}$$

if $G = \mathrm{Sp}(2k, \mathbf{C})$, $A_{i_j} = X_{i_j}$ *or* $X^{\mathrm{T}}_{i_j}$, *where* X^{T} *is the transpose of* X, *if* $G = \mathrm{O}(N)$ *and one may restrict to the case* $\nu \leq 2^N - 1$ *in both cases.*

Let $\sigma_n = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ and $\tau_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, and let $\mathrm{SU}(n, n)$ be the linear isometry group for the Hermitian form

$$|Z_1|^2 + |Z_2|^2 + \cdots + |Z_n|^2 - |Z_{n+1}|^2 - \cdots - |Z_{2n}|^2 \quad \text{over } \mathbf{C} .$$

The dual group for $\mathrm{Sp}(2k, \mathbf{C})$ is $\mathrm{SO}^*(2n) \equiv \{g \in \mathrm{SU}(n, n) | g^{\mathrm{T}}\sigma_n g = \sigma_n\}$, and the maximal compact subgroup of $\mathrm{SO}^*(2n)$ is

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} | A \in \mathrm{U}(n) \right\} \cong \mathrm{U}(n) .$$

The dual group for $\mathrm{O}(N)$ is $\mathrm{Sp}(2n, \Re) \equiv \{g \in \mathrm{SU}(n, n) | g^{\mathrm{T}}\tau_n g = \tau_n\}$, and $\mathrm{U}(n)$ is embedded in $\mathrm{Sp}(2n, \Re)$ as in $\mathrm{SO}^*(2n)$.

In this paper, we will construct the algebra of all K'-invariant polynomials of $G' \equiv \mathrm{SO}^*(2n)$ or $\mathrm{Sp}(2n, \Re)$, where K' is a block diagonal subgroup of G'.

2. Invariant polynomials of $\mathrm{SO}^*(2n)$

Let $(p_1, \ldots, p_m)$ be an m-tuple of positive integers such that $p_1 + \cdots + p_m = n$. Also, let $H \equiv \mathrm{GL}(2k, \mathbf{C})$, $G \equiv \mathrm{Sp}(2k, \mathbf{C})$, $G' \equiv \mathrm{SO}^*(2n)$, $\mathcal{G}' \equiv \mathrm{so}^*(2n)$, the Lie algebra of $\mathrm{SO}^*(2n)$, and K' is a block diagonal subgroup of G' consisting of matrices of the form

$$\begin{pmatrix} U_1 & & & & & \\ & \ddots & & & 0 & \\ & & U_m & & & \\ & & & \overline{U}_1 & & \\ & 0 & & & \ddots & \\ & & & & & \overline{U}_m \end{pmatrix}$$

where U_i is a $p_i \times p_i$ matrix in $U(p_i)$, for $1 \leq i \leq m$.

Let $\mathbf{C}^{n\times 2k}$ denote the complex vector space of all $n \times 2k$ matrices. Let $\mathbf{P} = \mathbf{P}(\mathbf{C}^{n\times 2k})$ denote the algebra of all polynomial functions on $\mathbf{C}^{n\times 2k}$. We partition $Z \in \mathbf{C}^{n\times 2k}$ into the following block form:

$$Z = \begin{bmatrix} Z_1 \\ \vdots \\ Z_m \end{bmatrix}, \qquad \text{where each } Z_i \in \mathbf{C}^{p_i \times 2k} .$$

We define an action of the direct product $H^{(1)} \times \cdots \times H^{(m)}$, where each $H^{(i)}$, $1 \le i \le m$, is a copy of H, on $\mathbf{C}^{n\times 2k}$ by

$$Z \cdot (h_1, \dots, h_m) = \begin{bmatrix} Z_1 h_1 \\ \vdots \\ Z_m h_m \end{bmatrix}, \qquad Z \in \mathbf{C}^{n\times 2k} .$$

This action gives rise to the actions of $H^{(1)} \times \cdots \times H^{(m)}$ and H on $\mathbf{P}$ via

$$(R(h_1, \dots, h_m)f)(Z) = f(Z \cdot (h_1, \dots, h_m))$$

and

$$(R(h)f)(Z) = f(Z \cdot h), \qquad h \equiv (h, \dots, h), \ \ h \in H .$$

The block diagonal subgroup $\mathrm{GL}(p_1, \mathbf{C}) \times \cdots \times \mathrm{GL}(p_m, \mathbf{C})$, the complexification of K', of $\mathrm{GL}(n, \mathbf{C})$ acts on $\mathbf{C}^{n\times 2k}$ via

$$(k_1, \dots, k_m) \cdot Z = \begin{bmatrix} k_1 Z_1 \\ \vdots \\ k_m Z_m \end{bmatrix}, \qquad k_i \in \mathrm{GL}(p_i, \mathbf{C}), \ \ 1 \le i \le m .$$

This action also gives rise to the action of $\mathrm{GL}(p_1, \mathbf{C}) \times \cdots \times \mathrm{GL}(p_m, \mathbf{C})$ on $\mathbf{P}$ via

$$(L(k_1, \dots, k_m)f)(Z) = f((k_1^{-1}, \dots, k_m^{-1}) \cdot Z)$$

for $(k_1, \dots, k_m) \in \mathrm{GL}(p_1, \mathbf{C}) \times \cdots \times \mathrm{GL}(p_m, \mathbf{C})$.

Let

$$R_{ij}^{(s)} = \sum_{\eta} Z_{\eta i} \frac{\partial}{\partial Z_{\eta j}}, \qquad 1 \le i, j \le 2k ,$$

where η ranges over all rows of the submatrix Z_s for a fixed index s, $1 \le s \le m$, of the matrix

$$Z = \begin{bmatrix} Z_1 \\ \vdots \\ Z_m \end{bmatrix} .$$

For each s, the operators $R_{ij}^{(s)}$, $1 \le i, j \le 2k$, form a basis for the Lie algebra $\mathscr{H}^{(s)} \equiv \mathrm{gl}(2k, \mathbf{C})$. We also obtain the infinitesimal operators corresponding to the standard basis of the Lie algebra of $\mathrm{Sp}(2k, \mathbf{C})$ as follows:

$$R_{\alpha\beta} = \sum_{\ell=1}^{n} \left(Z_{\ell\alpha} \frac{\partial}{\partial Z_{\ell\beta}} - Z_{\ell\beta+k} \frac{\partial}{\partial Z_{\ell\alpha+k}} \right),$$

$$R_{\alpha\beta+k} = \sum_{\ell=1}^{n} \left(Z_{\ell\alpha} \frac{\partial}{\partial Z_{\ell\beta+k}} + Z_{\ell\beta} \frac{\partial}{\partial Z_{\ell\alpha+k}} \right),$$

$$R_{\alpha+k\beta} = \sum_{\ell=1}^{n} \left(Z_{\ell\alpha+k} \frac{\partial}{\partial Z_{\ell\beta}} + Z_{\ell\beta+k} \frac{\partial}{\partial Z_{\ell\alpha}} \right), \qquad 1 \le \alpha, \beta \le k .$$

Now, the operators $R_{ij}^{(s)}$, $1 \leq i, j \leq 2k$, $1 \leq s \leq m$, generate the universal enveloping algebra $\mathscr{U}(\mathscr{H}^{(1)} \times \cdots \times \mathscr{H}^{(m)})$. It is not hard to show that in order to find the polynomial invariants in $\mathbf{S}(\mathscr{G}'^*)$ of a block diagonal subgroup K', it suffices to find the polynomial invariants in $\mathbf{S}(\Gamma^*)$, where $\Gamma \equiv \mathrm{so}(2n, \mathbf{C})$ is the complexification of $\mathrm{so}^*(2n)$, of the block diagonal subgroup $\mathrm{GL}(p_1, \mathbf{C}) \times \cdots \times \mathrm{GL}(p_m, \mathbf{C})$.

Let $s \in \Gamma$. Then s is of the form $s = \begin{bmatrix} Y & X \\ W & Q \end{bmatrix}$, where $[Y]$ is an $n \times n$ complex matrix, $[Q] = [-Y]^{\mathrm{T}}$, and $[W]$, $[X]$ are $n \times n$ skew-symmetric matrices. We partition s in the block matrices of the form:

$$s = \begin{bmatrix} [Y]_{11} & \cdots & [Y]_{1m} & [X]_{11} & \cdots & [X]_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ [Y]_{m1} & \cdots & [Y]_{mm} & [X]_{m1} & \cdots & [X]_{mm} \\ [W]_{11} & \cdots & [W]_{1m} & [Q]_{11} & \cdots & [Q]_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ [W]_{m1} & \cdots & [W]_{mm} & [Q]_{m1} & \cdots & [Q]_{mm} \end{bmatrix} = \begin{bmatrix} [Y]_{uv} & [X]_{uv} \\ [W]_{uv} & [Q]_{uv} \end{bmatrix},$$

where each $[Y]_{uv}$, $[X]_{uv}$, $[W]_{uv}$, $[Q]_{uv}$ is a $p_u \times p_v$ matrix, $1 \leq u, v \leq m$.

Recall that K' is a block diagonal subgroup of G'. Now, we can state our main theorem.

Theorem 4. *The algebra of all K'-invariant polynomials of $\mathbf{S}(\mathscr{G}'^*)$ is finitely generated by the constants and the trace of products of matrices of the following types:*

(1) $[Y]_{u_1u_2}[Y]_{u_2u_3}\cdots[Y]_{u_qu_{q+1}}$,
(2) $[X]_{\nu_1\nu_2}[W]_{\nu_2\nu_3}[X]_{\nu_3\nu_4}[W]_{\nu_4\nu_5}\cdots[X]_{\nu_r\nu_{r+1}}[W]_{\nu_{r+1}\nu_{r+2}}$, *and*
(3) $[X]_{t_1t_2}[Q]_{t_2t_3}\cdots[Q]_{t_st_{s+1}}[W]_{t_{s+1}t_{s+2}}$,

where if we multiply two different types together, then their indices must agree at the position of multiplication. Moreover, the first and the last indices of a product of matrices of these three types must be the same.

For example, we can form the product

$$\begin{aligned}&[Y]_{u_1u_2}\cdots[Y]_{u_qv_1}[X]_{v_1v_2}[W]_{v_2v_3}\cdots \\ &\qquad [X]_{v_{r-1}v_r}[W]_{v_rt_1}[X]_{t_1t_2}[Q]_{t_2t_3}\cdots[Q]_{t_st_{s+1}}[W]_{t_{s+1}u_1}.\end{aligned}$$

The proof of the theorem requires some preliminary results (cf. [KT1]) and several lemmas. Let $\mathscr{L}$ be a semisimple Lie algebra over a field of characteristic zero. Let $S \equiv S(\mathscr{L}^*)$ be the symmetric algebra of $\mathscr{L}$ and $\mathbf{U} \equiv \mathbf{U}(\mathscr{L})$ be the universal enveloping algebra of $\mathscr{L}$. Then it is well known that there exists the canonical filtration $\mathbf{U}_0 \subset \mathbf{U}_1 \subset \cdots \subset \mathbf{U}_n$ of $\mathscr{L}$ in $\mathbf{U}$. Also, we have the following commutative diagram:

$$\begin{array}{ccc} T^n & \xrightarrow{\psi_n} & U_n \\ {\scriptstyle \tau_n}\downarrow & & \downarrow{\scriptstyle \theta_n} \\ S^n & \xrightarrow[\varphi_n]{} & \Gamma^n \end{array}$$

where S^n denotes the nth symmetric power of $\mathscr{L}$, $T^n = \mathscr{L} \otimes \cdots \otimes \mathscr{L}$ (n times), $\Gamma^n \cong \mathbf{U}_n/\mathbf{U}_{n-1}$, τ_n, ψ_n, and θ_n are the corresponding canonical projections, and φ_n is the restriction of the canonical isomorphism of S on $\Gamma_0 \equiv \sum_{n\geq 0} \Gamma^n$ which is induced by the surjection $T = \sum_{n\geq 0} T^n \to \Gamma_0$ and implemented by the Poincaré-Birkhoff-Witt theorem. Also, it is well known that there exists a canonical bijection ϕ_n of S^n onto $\mathbf{U}_n$, termed the symmetrization, for $n \geq 0$. Now, let M be the adjoint group of $\mathscr{L}$ (cf. [Di]). We also have the following theorem (cf. [Go, Chapter 5, Tome 2]):

The canonical bijection ϕ of S on $\mathbf{U}$ (or ϕ_n of S^n onto $\mathbf{U}_n$) is an M-module isomorphism.

Let K be a closed and connected subgroup of M with the corresponding Lie algebra $\mathscr{K}$. An element $u \in \mathbf{U}$ is said to be $\mathscr{K}$-invariant if $Xu - uX = 0$ for all $X \in \mathscr{K}$. Similarly, we have the notion of K-invariant polynomial functions in S as mentioned in the introduction.

Let $R^{(s)}$ denote the $2k \times 2k$ matrix $(R_{ij}^{(s)})$, $1 \leq s \leq m$. Also, define

$$R^{(s)^*} = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix} R^{(s)^{\mathrm{T}}} \begin{bmatrix} 0 & -I_k \\ I_k & 0 \end{bmatrix}$$

for $1 \leq s \leq m$. Then we have

Lemma 5. *The algebra $\mathscr{V}$ of all $\mathscr{G}$-invariant differential operators of the universal enveloping algebra $\mathscr{U}(\mathscr{H}^{(1)} \times \cdots \times \mathscr{H}^{(m)})$ is finitely generated by the generalized Casimir operators $\mathrm{Tr}(A^{(i_1)} \cdots A^{(i_v)})$, where $A^{(i_j)} = R^{(i_j)}$ or $R^{(i_j)^*}$, and the constants.*

Proof. Let us first show that each differential operator of the form

$$\mathrm{Tr}(A^{(i_1)} \cdots A^{(i_s)}), \qquad A^{(i_j)} = R^{(i_j)} \text{ or } R^{(i_j)^*},$$

is $\mathscr{G}$-invariant. We now assume that $\mathscr{U}(\mathscr{H}^{(1)} \times \cdots \times \mathscr{H}^{(m)})$ contains a system of elements u_{ij}, $1 \leq i, j \leq 2k$, such that

(1) $[R_{\alpha\beta}, u_{ij}] = \delta_{\beta i} u_{\alpha j} - \delta_{\alpha j} u_{i\beta}$,
$[R_{\alpha\beta+k}, u_{ij}] = -\delta_{\alpha j} u_{i\beta+k} - \delta_{\beta j} u_{i\alpha+k}$ if $i, j \leq k$,
$[R_{\alpha+k\beta}, u_{ij}] = \delta_{\alpha i} u_{\beta+kj} + \delta_{\beta i} u_{\alpha+kj}$;

(2) $[R_{\alpha\beta}, u_{ij}] = \delta_{\beta+kj} u_{i\alpha+k} - \delta_{\alpha+ki} u_{\beta+kj}$,
$[R_{\alpha\beta+k}, u_{ij}] = \delta_{\beta+ki} u_{\alpha j} + \delta_{\alpha+ki} u_{\beta j}$ if $i, j > k$,
$[R_{\alpha+k\beta}, u_{ij}] = -\delta_{\beta+kj} u_{i\alpha} - \delta_{\alpha+kj} u_{i\beta}$;

(3) $[R_{\alpha\beta}, u_{ij}] = \delta_{\beta i} u_{\alpha j} + \delta_{\alpha+kj} u_{i\alpha+k}$,
$[R_{\alpha\beta+k}, u_{ij}] = 0$ if $j > k$, $i \leq k$,
$[R_{\alpha+k\beta}, u_{ij}] = -\delta_{\beta+kj} u_{i\alpha} - \delta_{\alpha+kj} u_{i\beta} + \delta_{i\alpha} u_{\beta+kj} + \delta_{i\beta} u_{\alpha+kj}$; and

(4) $[R_{\alpha\beta}, u_{ij}] = -\delta_{j\alpha} u_{i\beta} - \delta_{\alpha+ki} u_{\beta+kj}$,
$[R_{\alpha\beta+k}, u_{ij}] = \delta_{\beta+ki} u_{\alpha j} + \delta_{\alpha+ki} u_{\beta j} - \delta_{j\alpha} u_{i\beta+k} - \delta_{j\beta} u_{i\alpha+k}$ if $i > k$, $j \leq k$,
$[R_{\alpha+k\beta}, u_{ij}] = 0$.

Then clearly,

$$\begin{aligned}
\left[R_{\alpha\beta}, \sum_{i=1}^{2k} u_{ij}\right] &= \sum_{i=1}^{k}(\delta_{\beta i}u_{\alpha i} - \delta_{\alpha i}u_{i\beta}) + \sum_{i=k+1}^{2k}(\delta_{\beta+ki}u_{i\alpha+k} - \delta_{\alpha+ki}u_{\beta+ki}) \\
&= u_{\alpha\beta} - u_{\alpha\beta} + u_{\beta+k\alpha+k} - u_{\beta+k\alpha+k} = 0, \\
\left[R_{\alpha\beta+k}, \sum_{i=1}^{2k} u_{ij}\right] &= -\sum_{i=1}^{k}(\delta_{\alpha i}u_{i\beta+k} + \delta_{\beta i}u_{i\alpha+k}) + \sum_{i=k+1}^{2k}(\delta_{\beta+ki}u_{\alpha i} + \delta_{\alpha+ki}u_{\beta i}) \\
&= u_{\alpha\beta+k} - u_{\alpha\beta+k} + u_{\beta\alpha+k} - u_{\beta\alpha+k} = 0, \\
\left[R_{\alpha+k\beta}, \sum_{i=1}^{2k} u_{ij}\right] &= \sum_{i=1}^{k}(\delta_{\alpha i}u_{\beta+ki} + \delta_{\beta i}u_{\alpha+ki}) - \sum_{i=k+1}^{2k}(\delta_{\beta+ki}u_{i\alpha} + \delta_{\alpha+ki}u_{i\beta}) \\
&= u_{\alpha+k\beta} - u_{\alpha+k\beta} + u_{\beta+k\alpha} - u_{\beta+k\alpha} = 0 .
\end{aligned}$$

Let $\{u_{ij}\}$, $\{v_{ij}\}$ be two such systems. If $w_{ij} = u_{i1}v_{1j} + \cdots + u_{i2k}v_{2kj}$, then the w_{ij} form a system of the same type. Now, observe that $\{R^{(i_j)}_{\alpha\beta}\}$, $\{R^{(i_j)*}_{\alpha\beta}\}$, $1 \le \alpha, \beta \le 2k$, $1 \le i_j \le m$, are two such systems in $\mathscr{U}(\mathscr{H}^{(1)} \times \cdots \times \mathscr{H}^{(m)})$. Hence, by induction,

$$\operatorname{Tr}(A^{(i_1)} \cdots A^{(i_s)}) = \tilde{R}^{(i_1)}_{\alpha_1\alpha_2}\tilde{R}^{(i_2)}_{\alpha_2\alpha_3} \cdots \tilde{R}^{(i_s)}_{\alpha_s\alpha_1},$$

where

$$\tilde{R}^{(i_j)}_{\alpha_j\alpha_{j+1}} = R^{(i_j)}_{\alpha_j\alpha_{j+1}} \quad \text{if } A^{(i_j)} = R^{(i_j)}$$

or

$$\tilde{R}^{(i_j)}_{\alpha_j\alpha_{j+1}} = R^{(i_j)*}_{\alpha_j\alpha_{j+1}} \quad \text{if } A^{(i_j)} = R^{(i_j)*},$$

are indeed $\mathscr{G}$-invariant (notice that in the equation above, we use Einstein's convention). Now, let $\mathbf{S} = \mathbf{S}(\mathscr{H}^{(1)} \times \cdots \times \mathscr{H}^{(m)})$ and $\mathbf{U} = \mathscr{U}(\mathscr{H}^{(1)} \times \cdots \times \mathscr{H}^{(m)})$. From the previous discussion, we have the diagram:

$$\begin{tikzcd}
& U_k \arrow[d, "\theta_k"] \\
S^k \arrow[ur, "\phi_k"] \arrow[r, "\varphi_k"'] & \Gamma^k
\end{tikzcd}$$

where ϕ_k is the symmetrization and θ_k is the canonical projection. Let $d_{i_1\cdots i_k}$ denote the images in $\mathbf{U}$ of the polynomials $\operatorname{Tr}(A_{i_1} \cdots A_{i_k})$ by the symmetrization. Then by Theorem 3, $d_{i_1\cdots i_k}$ generates the subalgebra of all $\mathscr{G}$-invariant differential operators in $\mathbf{U}$. Now, for a fixed k-tuple $(i_1, \ldots, i_k)$ let $c_{i_1\cdots i_k} = \operatorname{Tr}(A^{(i_1)} \cdots A^{(i_k)})$. Then $\theta_k(d_{i_1\cdots i_k}) = \theta_k(c_{i_1\cdots i_k})$. Hence, $c_{i_1\cdots i_k} - d_{i_1\cdots i_k} \in \mathbf{U}_{k-1}$ and $c_{i_1\cdots i_k} - d_{i_1\cdots i_k}$ belong to the algebra of $\mathscr{G}$-invariant differential operators. For $k = 1$, we have $c_{i_1} = d_{i_1}$. By induction, we may assume that $c_{i_1\cdots i_q}$ generate the $\mathscr{G}$-invariant differential operators of degree $q \le k-1$. It follows that $d_{i_1\cdots i_k}$ can be expressed in terms of $c_{i_1\cdots i_k}$ and $c_{i_1\cdots i_q}$ for $q \le k-1$. This completes the proof of this theorem. Q.E.D.

Let

$$L_{\alpha\beta} = \sum_{\ell=1}^{2k} Z_{\alpha\ell}\frac{\partial}{\partial Z_{\beta\ell}}, \qquad P_{\alpha\beta} = \sum_{\ell=1}^{k}(Z_{\alpha\ell+k}Z_{\beta\ell} - Z_{\alpha\ell}Z_{\beta\ell+k}),$$

$$D_{\alpha\beta} = \sum_{\ell=1}^{k}(\frac{\partial^2}{\partial Z_{\alpha\ell+k}\partial Z_{\beta\ell}} - \frac{\partial^2}{\partial Z_{\alpha\ell}\partial Z_{\beta\ell+k}}), \qquad 1 \le \alpha, \beta \le n.$$

These operators form a basis for the Lie algebra Γ and generate the universal enveloping algebra $\mathscr{U}(\Gamma)$. Moreover, by the Poincaré-Birkhoff-Witt theorem, the ordered monomials in $L_{\alpha\beta}$, $D_{\alpha\beta}$, and $P_{\alpha\beta}$ form a basis for $\mathscr{U}(\Gamma)$. Amongst the operators $L_{\alpha\beta}$, we have the particular operators $L_{\alpha_p\beta_p}$, $p = p_1, \dots, p_m$, which correspond to the infinitesimal operators of the group $\mathrm{GL}(p_r, \mathbf{C})$, $1 \le r \le m$, acting on $\mathbf{P}$. They generate the Lie subalgebra $\mathrm{gl}(p_1, \mathbf{C})\times\cdots\times\mathrm{gl}(p_r, \mathbf{C})$. Now, consider the matrices

$$[L] = [L_{\alpha\beta}], \quad [P] = [P_{\alpha\beta}], \quad [D] = [D_{\alpha\beta}], \quad [E] = [E_{\alpha\beta}] = [-L_{\beta\alpha}].$$

We rewrite these matrices in the form

$$[L] = [[L]_{uv}], \quad [P] = [[P]_{uv}], \quad [D] = [[D]_{uv}], \quad [E] = [[E]_{uv}],$$

where $[L]_{uv}$, $[P]_{uv}$, $[D]_{uv}$, or $[E]_{uv}$ is a $p_u \times p_v$ matrix, $1 \le u, v \le m$.

Lemma 6. *In the universal enveloping algebra $\mathscr{U}(\Gamma)$, the constants and the trace of the product of the matrices of the forms:*

(1) $[L]_{u_1u_2}[L]_{u_2u_3}\cdots[L]_{u_qu_{q+1}}$,
(2) $[P]_{v_1v_2}[D]_{v_2v_3}[P]_{v_3v_4}[D]_{v_4v_5}\cdots[P]_{v_rv_{r+1}}[D]_{v_{r+1}v_{r+2}}$, *and*
(3) $[P]_{t_1t_2}[E]_{t_2t_3}\cdots[E]_{t_st_{s+1}}[D]_{t_{s+1}t_{s+2}}$,

with the same conditions as Theorem 4, *generate the same subalgebra $\mathscr{D}$ of differential operators.*

Proof. To prove this lemma, we need to express the differential operators in Lemma 5 in terms of the differential operators in Lemma 6 and vice versa. To illustrate the main idea of the proof of this lemma, we first consider differential operators of degree up to three. To avoid cumbersome sums in the traces of matrices, we adopt the Einstein convention of summing over repeated indices. To simplify the proof, we shall call the operators of the form in Lemma 6 left Casimir invariants and the operators of the form in Lemma 5 right Casimir invariants. We will use the obvious identity

$$\frac{\partial}{\partial z_{\alpha i}}z_{\beta_j} = \delta_{ij}\delta_{\alpha\beta} + z_{\beta_j}\frac{\partial}{\partial z_{\alpha i}}.$$

For a first-degree right Casimir invariant, we have $\mathrm{Tr}(R^{(s)*}) = \mathrm{Tr}(R^{(s)}) = R^{(s)}_{ii} = z_{\alpha i}(\partial/\partial z_{\alpha i}) = L_{\alpha\alpha} = \mathrm{Tr}([L_{ss}])$, for all $s = 1, \dots, m$, so that the assertion is certainly true. Recall that if $R^{(i)}_{\alpha\beta}$ denotes the $\alpha\beta$ entry of the matrix $R^{(i)}$, $1 \le i \le m$, then we have

$$R^{(i)}_{\alpha\beta} = z_{\eta\alpha}\frac{\partial}{\partial z_{\eta\beta}}, \qquad 1 \le \alpha, \beta \le 2k.$$

If $R^{(i)*}_{\alpha\beta}$ denotes the $\alpha\beta$ entry of the matrix $R^{(i)*}$, $1 \le i \le m$, then we have

$$R^{(i)*}_{\alpha\beta} = \begin{cases} z_{\eta\beta-k}\dfrac{\partial}{\partial z_{\eta\alpha-k}} & \text{if } k < \alpha,\ \beta \le 2k; \\ z_{\eta\beta+k}\dfrac{\partial}{\partial z_{\eta\alpha+k}}, & \text{if } k \ge \alpha,\ \beta \ge 1; \\ -z_{\eta\beta-k}\dfrac{\partial}{\partial z_{\eta\alpha+k}}, & \text{if } 1 \le \alpha \le k,\ 2k \ge \beta > k; \\ -z_{\eta\beta+k}\dfrac{\partial}{\partial z_{\eta\alpha-k}}, & \text{if } 1 \le \beta \le k,\ 2k \ge \alpha > k. \end{cases}$$

We are going to use the following notation throughout the proof:

$$\tilde{R}_{\alpha\beta} = z_{\eta\alpha}\frac{\partial}{\partial z_{\eta\beta}} \quad \text{if } \tilde{R}_{\alpha\beta} \text{ is the } \alpha\beta \text{ entry of } R^{(i)} \text{ for some } 1 \le i \le m,$$

or

$$\tilde{R}_{\alpha\beta} = \pm z_{\eta\beta+k}\frac{\partial}{\partial_{\eta\alpha\pm k}} \quad \text{if } \tilde{R}_{\alpha\beta} \text{ is the } \alpha\beta \text{ entry of } R^{(i)*} \text{ for some } 1 \le i \le m.$$

Let $R_i \in \{R^{(1)}, \ldots, R^{(m)}\}$, $i = 1, 2, 3$. Let us first consider the second-degree Casimir invariants. We have several cases to consider.

Case 1. Consider the right Casimir invariant

$$\begin{aligned} \mathrm{Tr}(R_1R_2^*) = \tilde{R}_{ij}\tilde{R}_{ji} &= z_{\alpha i}\frac{\partial}{\partial z_{\alpha j}} \pm \left(z_{\beta i\pm k}\frac{\partial}{\partial z_{\beta j\pm k}}\right) \\ &= \pm\left(z_{\alpha i}\left(z_{\beta i\pm k}\frac{\partial}{\partial z_{\alpha j}} - \delta_{\alpha\beta}\delta_{ji\pm k}\right)\frac{\partial}{\partial z_{\beta j\pm k}}\right) \\ &= \pm\left(z_{\alpha i}z_{\beta i\pm k}\frac{\partial}{\partial z_{\beta j\pm k}}\frac{\partial}{\partial z_{\alpha j}}\right) - \pm\left(z_{\alpha i}\frac{\partial}{\partial z_{\alpha i}}\right) \\ &= \pm\mathrm{Tr}(R_1) \pm \left(z_{\alpha i}z_{\beta i\pm k}\frac{\partial}{\partial z_{\beta j\pm k}}\frac{\partial}{\partial z_{\alpha j}}\right). \end{aligned}$$

Therefore, $\mathrm{Tr}(R_1R_2^*)$ can be expressed in terms of the sum of a left Casimir invariant $\mathrm{Tr}(PD)$ and some first-degree invariants, and vice versa.

Case 2. Consider the right Casimir invariant $\mathrm{Tr}(R_1^*R_2) = \tilde{R}_{ij}\tilde{R}_{ji}$. A similar computation as in Case 1 shows that $\mathrm{Tr}(R_1^*R_2)$ can be expressed in terms of the sum of a left Casimir invariant $\mathrm{Tr}(PD)$ and some first degree invariants, and vice versa.

Case 3. Following the proof of Lemma 3.2 in [KT1], the second-degree Casimir invariants $\mathrm{Tr}(R_1R_2)$ and $\mathrm{Tr}(R_1^*R_2^*)$ can be expressed in terms of the sum of a left Casimir invariant $\mathrm{Tr}(LL)$ and some first-degree left Casimir invariants, and vice versa.

The above cases have exhausted all possible second-degree right Casimir invariants. Let us now consider the third-degree Casimir invariants. There are also several cases to consider.

Case 1. Consider the right Casimir invariant

$$\begin{aligned}
\operatorname{Tr}(R_1R_2R_3^*) = \tilde{R}_{ij}\tilde{R}_{jp}\tilde{R}_{pi} &= z_{\alpha i}\frac{\partial}{\partial z_{\alpha j}} z_{\beta j}\frac{\partial}{\partial z_{\beta p}} \pm \left(z_{\gamma i\pm k}\frac{\partial}{\partial z_{\gamma p\pm k}}\right) \\
&= z_{\alpha i}\frac{\partial}{\partial z_{\alpha j}} z_{\beta j}\left(z_{\gamma i\pm k}\frac{\partial}{\partial z_{\beta p}} - \delta_{\alpha\beta}\delta_{pi\pm k}\right)\left(\pm\frac{\partial}{\partial z_{\gamma p\pm k}}\right) \\
&= z_{\alpha i}\frac{\partial}{\partial z_{\alpha j}} z_{\beta j}\frac{\partial}{\partial z_{\beta i}} + z_{\alpha i}\frac{\partial}{\partial z_{\alpha j}} z_{\beta j} z_{\gamma i\pm k}\frac{\partial}{\partial z_{\beta p}}\left(\pm\frac{\partial}{\partial z_{\gamma p\pm k}}\right) .
\end{aligned}$$

The first expression above is equal to $\operatorname{Tr}(R_1R_2)$, hence we only need to consider the second expression. Now,

$$\begin{aligned}
z_{\alpha i}\frac{\partial}{\partial z_{\alpha j}} z_{\beta j} z_{\gamma i\pm k}\frac{\partial}{\partial z_{\beta p}}\left(\pm\frac{\partial}{\partial z_{\gamma p\pm k}}\right) &= z_{\alpha i}\frac{\partial}{\partial z_{\alpha j}} z_{\gamma i\pm k} z_{\beta j}\frac{\partial}{\partial z_{\beta p}}\left(\pm\frac{\partial}{\partial z_{\gamma p\pm k}}\right) \\
= z_{\alpha i}\left(-\delta_{\alpha\gamma}\delta_{ji\pm k} + z_{\gamma i\pm k}\frac{\partial}{\partial z_{\alpha j}}\right) z_{\beta j}\frac{\partial}{\partial z_{\beta p}}\left(\pm\frac{\partial}{\partial z_{\gamma p\pm k}}\right)& \\
= \pm\left(z_{\gamma i} z_{\beta j\pm k}\frac{\partial}{\partial z_{\beta p}}\frac{\partial}{\partial z_{\gamma p\pm k}}\right) + z_{\alpha i} z_{\gamma i\pm k}\frac{\partial}{\partial z_{\alpha j}} z_{\beta j}\frac{\partial}{\partial z_{\beta p}}\left(\pm\frac{\partial}{\partial z_{\gamma p\pm k}}\right) .&
\end{aligned}$$

The first expression above can be handled as in previous cases. Therefore, we only need to consider the second expression:

$$\begin{aligned}
z_{\alpha i} z_{\gamma i\pm k}\frac{\partial}{\partial z_{\alpha j}} z_{\beta j}\frac{\partial}{\partial z_{\beta p}}\left(\pm\frac{\partial}{\partial z_{\gamma p\pm k}}\right) &= z_{\alpha i} z_{\gamma i\pm k}\frac{\partial}{\partial z_{\alpha j}} z_{\beta j}\left(\pm\frac{\partial}{\partial z_{\gamma p\pm k}}\right)\frac{\partial}{\partial z_{\beta p}} \\
= z_{\alpha i} z_{\gamma i\pm k}\frac{\partial}{\partial z_{\alpha j}} \pm \left(-\delta_{\beta\gamma}\delta_{jp\pm k} + \frac{\partial}{\partial z_{\gamma p\pm k}} z_{\beta j}\right)\frac{\partial}{\partial z_{\beta p}}& \\
= z_{\alpha i} z_{\gamma i\pm k}\frac{\partial}{\partial z_{\gamma j\pm k}}\frac{\partial}{\partial z_{\alpha j}} + z_{\alpha i} z_{\gamma i\pm k}\frac{\partial}{\partial z_{\alpha j}} \pm \left(\frac{\partial}{\partial z_{\gamma p\pm k}} z_{\beta j}\right)\frac{\partial}{\partial z_{\beta p}} .&
\end{aligned}$$

As before, by induction, we only need to consider the second expression:

$$\begin{aligned}
z_{\alpha i} z_{\gamma i\pm k}\frac{\partial}{\partial z_{\alpha j}} \pm \left(\frac{\partial}{\partial z_{\gamma p\pm k}} z_{\beta j}\right)\frac{\partial}{\partial z_{\beta p}} &= z_{\alpha i} z_{\gamma i\pm k} \pm \left(\frac{\partial}{\partial z_{\gamma p\pm k}}\right)\frac{\partial}{\partial z_{\alpha j}} z_{\beta j}\frac{\partial}{\partial z_{\beta p}} \\
= z_{\alpha i} z_{\gamma i\pm k} \pm \left(\frac{\partial}{\partial z_{\gamma p\pm k}}\right)\frac{\partial}{\partial z_{\alpha j}}\left(\frac{\partial}{\partial z_{\beta p}} z_{\beta j} - q_\beta\delta_{jp}\right)& \\
= \pm q_\beta\left(z_{\alpha i} z_{\gamma i\pm k}\frac{\partial}{\partial z_{\gamma p\pm k}}\frac{\partial}{\partial z_{\alpha p}}\right) + z_{\alpha i} z_{\gamma i\pm k} \pm \left(\frac{\partial}{\partial z_{\gamma p\pm k}}\right)\frac{\partial}{\partial z_{\alpha j}}\frac{\partial}{\partial z_{\beta p}} z_{\beta j} ,&
\end{aligned}$$

where q_β is the cardinality of the range of β.

Once again, we only need to consider the second expression:

$$\begin{aligned}
z_{\alpha i} z_{\gamma i\pm k} \pm \left(\frac{\partial}{\partial z_{\gamma p\pm k}}\right)\frac{\partial}{\partial z_{\alpha j}}\frac{\partial}{\partial z_{\beta p}} z_{\beta j} &= z_{\alpha i} z_{\gamma i\pm k} \pm \left(\frac{\partial}{\partial z_{\gamma p\pm k}}\right)\frac{\partial}{\partial z_{\beta p}}\frac{\partial}{\partial z_{\alpha j}} z_{\beta j} \\
= z_{\alpha i} z_{\gamma i\pm k} \pm \left(\frac{\partial}{\partial z_{\gamma p\pm k}}\right)\frac{\partial}{\partial z_{\beta p}}\left(z_{\beta j}\frac{\partial}{\partial z_{\alpha j}} - \delta_{\alpha\beta}\right)& \\
= \pm\left(z_{\alpha i} z_{\gamma i\pm k}\frac{\partial}{\partial z_{\gamma p\pm k}}\frac{\partial}{\partial z_{\alpha p}}\right) \pm \left(z_{\alpha i} z_{\gamma i\pm k}\frac{\partial}{\partial z_{\gamma p\pm k}}\frac{\partial}{\partial z_{\beta p}}\right) z_{\beta j}\frac{\partial}{\partial z_{\alpha j}} .&
\end{aligned}$$

What we have shown in this case is $\mathrm{Tr}(R_1 R_2 R_3^*)$ can be expressed in terms of the sum of a left Casimir invariant $\mathrm{Tr}(PDL)$ and some lower degree invariants, and vice versa.

Case 2. Let us consider the right Casimir invariant

$$\mathrm{Tr}(R_1 R_2^* R_3^*) = z_{\alpha i}\frac{\partial}{\partial z_{\alpha j}} \pm \left(z_{\beta p\pm k}\frac{\partial}{\partial z_{\beta j\pm k}}\right) \pm \left(z_{\gamma i\pm k}\frac{\partial}{\partial z_{\gamma p\pm k}}\right).$$

Then a similar computation as in Case 1 shows that $\mathrm{Tr}(R_1 R_2^* R_3^*)$ can be expressed in terms of the sum of a left Casimir invariant $\mathrm{Tr}(PED)$ and some lower degree invariants, and vice versa.

By iteration, it is easy to see that any other third-degree right Casimir invariant can be expressed in terms of the sum of a third-degree left Casimir invariant and some lower degree invariants, and vice versa. As a result, the lemma is true up to degree three. Now, by the same technique, it is clear that the lemma can be proved by induction in the same fashion as in [Le] on the degree of the right Casimir invariants, and vice versa. Hence, the proof of the lemma is now complete (see [Le] for details). Q.E.D.

Let $\mathscr{K}'$ be the Lie algebra of K'. Then we have the following

Lemma 7. *The algebra of all $\mathscr{K}'$-invariant differential operators in $\mathscr{U}(\mathscr{G}')$ is finitely generated by the same elements of the form as defined in Lemma* 6 *and the constants.*

Proof. By an argument similar to Lemma 5, the differential operators in Lemma 6 are indeed $\mathscr{K}'$-invariant. To complete the proof of this lemma, we only need to show there is a one-to-one correspondence between $\mathscr{G}$-invariant differential operators and $\mathscr{K}'$-invariant differential operators. This follows from

Lemma 8 (cf. [Ho2, Ho3, LT]). *Let $(M^i) = (M_1^i, \dots, M_{p_i}^i, 0, \dots, 0)$ be a $2k$-tuple of integers such that the condition $M_1^i \geq \cdots \geq M_{p_i}^i \geq 0$, for $1 \leq i \leq m$, is satisfied. Let μ' be the irreducible representation of K' indexed by the n-tuple of integers of the form:*

$$(M_1^1, \dots, M_{p_1}^1, M_1^2, \dots, M_{p_m}^m).$$

Let μ denote the tensor product of representations of $\mathrm{GL}(2k, \mathbf{C}) \times \cdots \times \mathrm{GL}(2k, \mathbf{C})$ *indexed by (M^i), $1 \leq i \leq m$. Let γ be an irreducible representation of G. There is one-to-one correspondence between the irreducible representation of G and the irreducible holomorphic discrete series representation labelled by the Blattner's parameter γ' of G'. Then the multiplicity* $\dim \mathrm{Hom}_{k'}(\mu', \gamma')$ *of the irreducible representation μ of K' in the restriction to K' of the irreducible representation γ' of G' is equal to the multiplicity* $\dim \mathrm{Hom}_G(\gamma, \mu)$ *of the irreducible representation γ of G in the Kronecker product representation μ of G.*

Lemma 9. *A polynomial function $f \in \mathbf{S}(\mathscr{G}'^*)$ is K'-invariant iff the corresponding differential operator $f \in \mathscr{U}(\mathscr{G}')$, by the map ϕ, is $\mathscr{K}'$-invariant.*

Proof. First observe that the canonical bijection ϕ is also a K'-module isomorphism. An argument similar to Lemma 5 shows that the canonical isomorphism ϕ carries the $\mathscr{K}'$-invariant differential operators of $\mathscr{U}(\mathscr{G}')$ to the K'-invariant polynomials of $\mathbf{S}(\mathscr{G}'^*)$. Q.E.D.

Theorem 4 now follows from Lemmas 5 and 7–9.

3. Invariant polynomials of $\mathrm{Sp}(2n, \mathfrak{R})$

In this section, let $H = \mathrm{GL}(N, \mathbf{C})$, $G = \mathrm{O}(N)$, $G' = \mathrm{Sp}(2n, \mathfrak{R})$, $\mathscr{G}' = \mathrm{sp}(2n, \mathfrak{R})$, the Lie algebra of G', and Γ be the complexification of $\mathscr{G}'$. Let $(p_1, \dots, p_m)$ be an m-tuple of positive integers such that $p_1 + \cdots + p_m = n$. Let K' be a block diagonal subgroup as before. Let $\mathbb{C}^{n\times N}$ be a $n \times N$ complex matrix. Then we have the same actions of H, G, and $\mathrm{GL}(N, \mathbf{C}) \times \cdots \times \mathrm{GL}(N, \mathbf{C})$ on $\mathbf{C}^{n\times N}$ and these actions give rise to the representations of these groups on $\mathbf{P}(\mathbf{C}^{n\times N})$.

Let $s \in \Gamma$ and write s as $s = \begin{bmatrix} Y & X \\ W & Q \end{bmatrix}$, where $[Y]$ is an $n\times n$ complex matrix, $[Q] = [-Y]^{\mathrm{T}}$, and $[X]$, $[W]$ are $n \times n$ symmetric matrices. We partition s into the block matrices as in the previous section. We now state the main result of this section.

Theorem 10. *The algebra of all K'-invariant polynomials of $\mathbf{S}(\mathscr{G}')$ is finitely generated by the constants and the trace of products of matrices of the following types*:

(1) $[Y]_{u_1u_2}[Y]_{u_2u_3}\cdots[Y]_{u_qu_{q+1}}$,
(2) $[X]_{\nu_1\nu_2}[W]_{\nu_2\nu_3}[X]_{\nu_3\nu_4}[W]_{\nu_4\nu_5}\cdots[X]_{\nu_r\nu_{r+1}}[W]_{\nu_{r+1}\nu_{r+2}}$, *and*
(3) $[X]_{t_1t_2}[Q]_{t_2t_3}\cdots[Q]_{t_st_{s+1}}[W]_{t_{s+1}t_{s+2}}$,

where if we multiply two different types together, then their indices must agree at the position of multiplication. Moreover, the first and the last indices of a product of matrices of these three types must be the same.

The proof of this theorem is similar to that of Theorem 4 (see [Le] for details).

4. Conclusion

In the course of the proof of our theorems, we introduced the generalized Casimir operators. These generalized Casimir operators are very important in physics. In [KT2], Klink and Ton-That have used these operators to solve the multiplicty problem in the tensor product of irreducible representations of $\mathrm{GL}(N, \mathbf{C})$. Their method can be used to solve the multiplicity problem in tensor products of irreducible representations of $\mathrm{Sp}(2k, \mathbf{C})$. This is carried out in detail in [Le].

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