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CHARACTER VALUES AT INVOLUTIONS

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ABSTRACT. If χ_1, χ_2, χ_3 are irreducible characters of a finite group G satisfying $\int_G \chi_1\chi_2\chi_3 \neq 0$ and σ is an involution in G, then the proportions of -1's among the eigenvalues of the corresponding representations at σ are the sides of a triangle on a sphere of circumference 2.

For an involution σ of a finite group G and an irreducible complex representation R of G, denote by q the proportion of -1's among the eigenvalues of $R(\sigma)$. The objects of this note are to prove the bounds

$$\frac{1}{h} \leq q \leq 1 - \frac{1}{h}, \quad \text{unless } q = 0 \text{ or } 1, \tag{1}$$

where h is the index of the centralizer C of σ, and then, refining (1), to prove "spherical triangle" inequalities

$$q_i + q_j + q_k \leq 2, \qquad q_i \leq q_j + q_k, \quad \text{etc.}, \tag{2}$$

for each triple R_i, R_j, R_k for which

$$R_i \otimes R_j \otimes R_k | G \tag{3}$$

contains the principal representation. Finally, we show the equivalence of:

(a) $q = 1 - \frac{1}{h}$ (or $\frac{1}{h}$);

(b) R is induced by the unique irreducible representation S in $R|C$ whose restriction to $\langle\sigma\rangle$ is a multiple of the principal (nonprincipal) representation; and

(c) there is equality in one of the first two (last two) inequalities in (2) with $R_i = R$ and all R_j, R_k satisfying (3).

Proof of (1). Let R represent G on a space V of dimension f with character χ. Let T be one of the two irreducible representations of $\langle\sigma\rangle$, and let V_T be the T-isotypic subspace of V. If $V_T \neq 0$, then $V = \sum_{\tau\in G} \tau V_T$ since R is irreducible. Since $\tau V_T = V_{T^\tau}$ depends only on τ mod C, it follows that

$$V = \sum_{\tau \bmod C} \tau V_T, \tag{4}$$

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so $f \leq he$ where e is the dimension of V_T. Since $e = (1-q)f$ or qf according as T is principal or not, the inequalities in (1) follow from hypotheses $V_T \neq 0$ in the two cases.

Proof. (a) $\Leftrightarrow$ (b). By the above argument, $q = 1 - \frac{1}{h}$ (or $\frac{1}{h}$) is equivalent to a direct sum in (4) with T principal (or nonprincipal). Let S be the representation of C on V_T. A direct sum in (4) means that R is induced by S, which must then be the unique irreducible representation in $R|C$ whose restriction to $\langle\sigma\rangle$ is a multiple of T. Conversely, if R is induced by such an S, then

$$\chi(\sigma) = \pm e \mp (f-e) = \mp f\left(1 - \frac{2}{h}\right),$$

giving

$$q = \frac{1}{2}\left(1 - \frac{\chi(\sigma)}{f}\right) = 1 - \frac{1}{h} \left(\text{or } \frac{1}{h}\right)$$

for T principal (or nonprincipal).

Proof. (3) $\Rightarrow$ (2). Put

$$P = \int_G R_i \otimes R_j \otimes R_k, \qquad c = c_{ijk} = \int_G \chi_i \chi_j \chi_k, \tag{5}$$

and for irreducible representations T_i, T_j, T_k of $\langle\sigma\rangle$ put

$$P' = \int_{\langle\sigma\rangle}\int_{\langle\sigma\rangle}\int_{\langle\sigma\rangle} (\psi_i \otimes \psi_j \otimes \psi_k)(R_i \otimes R_j \otimes R_k), \tag{6}$$

where the ψ's are the characters of the T's. Relative to a $G\times G\times G$ invariant positive definite Hermitian form on $V_i \otimes V_j \otimes V_k$, P and P' are selfadjoint idempotents, the orthogonal projections of $V_i \otimes V_j \otimes V_k$ onto the subspace of G-invariants and the subspace $(V_i)_{T_i} \otimes (V_j)_{T_j} \otimes (V_k)_{T_k}$ respectively. It follows that

$$\operatorname{tr}(P'P) = \operatorname{tr}((P'P)(P'P)^*) \geq 0. \tag{7}$$

From (5) and (6), $\operatorname{tr}(P'P)$ is

$$\int_{\langle\sigma\rangle}\int_{\langle\sigma\rangle}\int_{\langle\sigma\rangle} \psi_i(\sigma_i)\psi_j(\sigma_j)\psi_k(\sigma_k) \int_G \chi_i(\sigma_i\tau)\chi_j(\sigma_j\tau)\chi_k(\sigma_k\tau).$$

According as σ_i, σ_j, σ_k are the same $(1, 1, 1$ or $\sigma, \sigma, \sigma)$ or not, e.g., with $\sigma_i \neq \sigma_j = \sigma_k$, the inner integral is c or, e.g., $c\chi_i(\sigma)/f_i$. Choosing T_i, T_j, T_k all principal or only T_i principal gives

$$\operatorname{tr}(P'P) = \frac{c}{4}\left(1 + \frac{\chi_i(\sigma)}{f_i} \pm \frac{\chi_j(\sigma)}{f_j} \pm \frac{\chi_k(\sigma)}{f_k}\right). \tag{8}$$

Assuming $c \neq 0$, which is (3), the first two inequalities in (2) follow from (7) and (8) using $\chi(\sigma)/f = 1 - 2q$. The last two inequalities in (2) follow similarly.

Proof. (3) $\Rightarrow$ (2) implies (1) and (a) $\Leftrightarrow$ (c). From the orthogonality relations it follows that

$$\sum_{j,k} c_{ijk} f_j f_k = |G| f_i ,$$

$$\sum_{j,k} c_{ijk} \chi_j^2(\sigma)/f_j = |C| f_i ,$$

$$\sum_{j,k} c_{ijk} \chi_j(\sigma)\chi_k(\sigma) = |C| \chi_i(\sigma).$$

In terms of $a_{ijk} = c_{ijk} f_j f_k / f_i$ and $x_i = \chi_i(\sigma)/f_i$, these equations are

$$\sum_{j,k} a_{ijk} = |G| , \qquad \sum_{j,k} a_{ijk} x_j^2 = |C| , \qquad \sum_{j,k} a_{ijk} x_j x_k = |C| x_i .$$

Using $a_{ijk} \geq 0$ and $1 \pm x_i \geq |x_j \pm x_k|$ for $a_{ijk} \neq 0$, it follows that

$$(1 \pm x_i)^2 |G| \geq \sum_{j,k} a_{ijk} (x_j \pm x_k)^2 = 2(1 \pm x_i)|C| ,$$

which implies $1 \pm x_i \geq \frac{2}{h}$ for $x_i \neq \mp 1$, which is (1) again. It also follows that (a) is equivalent to $1 \pm x_i = |x_j \pm x_k|$ for all j, k with $a_{ijk} \neq 0$, i.e., for $+$ to equality in the first or second of the inequalities in (2) for all j, k with $c_{ijk} \neq 0$, and for $-$ to equality in the third or fourth of the inequalities in (2) for all j, k with $c_{ijk} \neq 0$.

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