PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 4, April 1994

STRONGLY EXPOSED POINTS IN LEBESGUE-BOCHNER FUNCTION SPACES

ZHIBAO HU AND BOR-LUH LIN

(Communicated by Dale Alspach)

ABSTRACT. It is a result of Peter Greim that if f is a strongly exposed point of the unit ball of Lebesgue-Bochner function space $L^p(\mu, X)$, $1 < p < \infty$, then f is a unit vector and $f(t)/\|f(t)\|$ is a strongly exposed point of the unit ball of X for almost all t in the support of f.

We prove that the converse is also true.

Throughout we assume that X is a Banach space, $1 < p, q < \infty$ with $1/p + 1/q = 1$, and (Ω, Σ, μ) is a positive measure space. We use S_X and B_X to denote the unit sphere and the unit ball in X respectively and use μ^* to denote the outer measure associated with μ. For a subset K of X we use str-exp$K$ to denote the set of strongly exposed points of K. Recall that x is a *strongly exposed point* of K if $x \in K$ and there exists x^* in X^* which strongly exposes K at x, that is, $\sup x^*(K) = x^*(x)$ and, whenever $\{x_n\} \subset K$ and $\lim_n x^*(x_n) = x^*(x)$, $\lim_n x_n = x$.

Johnson [J] and Greim [G1, G2] studied the strongly exposed points in Lebesgue-Bochner function spaces $L^p(\mu, X)$. In [G1, G2], it is shown that, for $f \in$ str-exp$B_{L^p(\mu, X)}$, it is necessary and, in case X is smooth, is also sufficient that, for almost all t in the support of f (supp$f$), $f(t)/\|f(t)\| \in$ str-expB_X and $\|f\| = 1$. The purpose of this note is to show that these conditions are sufficient in general (Theorem 7).

Strongly exposed points can be described in terms of slices. For $K \subset X$, the slice of K determined by the functional x^* in X^* and $\delta > 0$ is the subset of K given by

$$S(x^*, K, \delta) = \{x \in K: x^*(x) > \sup x^*(K) - \delta\}.$$

It is obvious that x^* strongly exposes K at x if and only if for any $\varepsilon > 0$ there is $\delta > 0$ such that $\operatorname{diam} S(x^*, K, \delta) < \varepsilon$ and $x \in S(x^*, K, \delta)$, where $\operatorname{diam} S(x^*, K, \delta)$ is the diameter of the set $S(x^*, K, \delta)$.

The key to our discussion is the following characterization of strongly exposed points "exclusively" in terms of slices (i.e., it does not use strongly exposing functionals): an element x in K is a strongly exposed point of K if and only if there exist $\varepsilon_n > 0$ and $\delta_n > 0$, with $\lim_n \varepsilon_n = 0$ and $\lim_n \delta_n = 0$, and

Received by the editors April 10, 1992 and, in revised form, July 21, 1992.
1991 *Mathematics Subject Classification*. Primary 46B20, 46E40.
Key words and phrases. Lebesgue-Bochner function space, strongly exposed point, slice.

a bounded sequence $\{x_n^*\} \subset X^*$, such that x belongs to each $S(x_n^*, K, \delta_n)$ and $\operatorname{diam} S(x_n^*, K, \delta_m) \leq \varepsilon_m$ for $n \geq m \geq 1$. Lemma 2 is a variation of this observation for Lebesgue-Bochner function spaces.

Lemma 1. *Let $f \in L^p(\mu)$ such that $f(t) \geq 0$ for all t in Ω. If $\{f_\lambda\}$ is a net in $L^p(\mu, X)$ such that $\{\|f_\lambda(\cdot)\|\}$ converges to f in $L^p(\mu)$ then there is a net $\{g_\lambda\}$ in $L^p(\mu, X)$ such that $\|g_\lambda(t)\| = f(t)$, for all t and λ, and $\lim_\lambda \|f_\lambda - g_\lambda\| = 0$.*

Proof. Pick x in S_X. For each λ, let $E_\lambda = \{t \colon t \in \Omega \text{ and } f_\lambda(t) = 0\}$ and define

$$g_\lambda(t) = \begin{cases} f(t)x, & t \in E_\lambda, \\ f(t) f_\lambda(t)/\|f_\lambda(t)\|, & t \notin E_\lambda. \end{cases}$$

Then $\|g_\lambda(t)\| = f(t)$ and $\|f_\lambda(t) - g_\lambda(t)\| = |\|f_\lambda(t)\| - f(t)|$ for all t and λ. Hence, $g_\lambda \in L^p(\mu, X)$ and $\lim_\lambda \|f_\lambda - g_\lambda\| = \lim_\lambda \|\|f_\lambda(\cdot)\| - f\| = 0$. Q.E.D.

For each $f \in L^p(\mu, X)$, we denote by $S(f)$ the set

$$\{g \colon g \in L^p(\mu, X) \text{ and } \|g(t)\| = \|f(t)\| \text{ for all } t \text{ in } \Omega\}.$$

Lemma 2. *Suppose $f \in S_{L^p(\mu, X)}$. Then the following are equivalent.*

(1) $f \in \text{str-exp}\, B_{L^p(\mu, X)}$.

(2) *There exist $\varepsilon_n > 0$ and $\delta_n > 0$, with $\lim_n \varepsilon_n = 0$ and $\lim_n \delta_n = 0$, and $\{T_n\}$ in $B_{L^p(\mu, X)^*}$ satisfying*

$T_n(f) > 1 - \delta_n$, and for $m \leq n$, if $g \in S(f)$ and $T_n(g) > 1 - \delta_m$, then $\|f - g\| \leq \varepsilon_m$.

Moreover, if (2) is true, then every weak cluster point of $\{T_n\}$ strongly exposes $B_{L^p(\mu, X)}$ at f.*

Proof. It is obvious that (1) implies (2). To prove the converse is true, let T be a weak* cluster point of $\{T_n\}$. Then $T(f) = 1$ and $\|T\| \leq 1$. Thus

$$\|T\| = 1 = T(f) = \sup T(B_{L^p(\mu, X)}).$$

Suppose $\{f_n\}$ is a sequence in $B_{L^p(\mu, X]}$ such that $\lim_n T(f_n) = 1$. Then $\lim_n T(f_n + f) = 2$. It follows that, in $L^p(\mu)$, $\lim_n \|\|f_n(\cdot)\| + \|f(\cdot)\|\| = 2$. By the uniform rotundity of $L^p(\mu)$, we have $\lim_n \|f_n(\cdot)\| = \|f(\cdot)\|$ in $L^p(\mu)$. Thus by Lemma 1 there is a sequence $\{g_n\}$ in $S(f)$ such that $\lim_n \|f_n - g_n\| = 0$. It is obvious that $\lim_n T(g_n) = 1$. For any $\varepsilon > 0$, choose $m \geq 1$ such that $\varepsilon_m < \varepsilon$. There is $n_1 > m$ such that for $k \geq n_1$ we have $T(g_k) > 1 - \delta_m$ and $\delta_k < \delta_m$. For each $k \geq n_1$, since T is a weak* cluster point of $\{T_n\}$ and $T(g_k) > 1 - \delta_m$, there is $n \geq n_1$ such that $T_n(g_k) > 1 - \delta_m$. Thus $\|f - g_k\| \leq \varepsilon_m < \varepsilon$. Hence, $\lim_n \|g_n - f\| = 0$. It follows that $\lim_n \|f_n - f\| = 0$. Therefore, T strongly exposes $B_{L^p(\mu, X)}$ at f. Q.E.D.

In the proof of Lemma 4, we will construct a sequence $\{T_n\}$ like the one described in Lemma 2, but our $\{T_n\}$ will be from $L^q(\mu, X^*)$, which is naturally a subspace of $L^p(\mu, X)^*$. Note that for any $\varphi \in L^q(\mu, X^*)$ and $h \in L^p(\mu, X)$ the action of φ on h is given by $(\varphi, h) = \int_\Omega (\varphi(t), h(t))\, d\mu(t)$ [DU]. For the proof of Lemma 4 we need

Lemma 3. *Suppose (Ω, Σ, μ) is a probability space and $f \in B_{L^p(\mu, X)}$ with $f(\Omega) \subset B_X$. Let $0 < \delta < 1$ and $g \colon \Omega \to S_{X^*}$. If there are $\{t_n\} \subset \Omega$, $\{F_n\} \subset \Sigma$, and $r > 0$ satisfying:*

(1) $\operatorname{diam} S(g(t_n), B_X, r) < \delta$ *and* $f(F_n) \subset S(g(t_n), B_X, r)$, *and*

(2) $F_n \cap F_m = \varnothing$ *for* $n \neq m$ *and* $\mu(\bigcup_n F_n) > 1 - 2\delta/3$,

then, for any measurable functions $\varphi\colon \Omega \to B_{X^*}$ *and* $h\colon \Omega \to B_X$ *with* $\varphi|_{\bigcup_n F_n} = \sum_n g(t_n)\chi_{F_n}$ *and* $\int_\Omega (\varphi(t), h(t))\, d\mu(t) \ge 1 - r\delta/3$, *we have* $\|h - f\| < 3\delta^{1/p}$.

Proof. Let $E_n = \{t\colon t \in F_n \text{ and } h(t) \notin S(g(t_n), B_X, r)\}$. Then $\|h(t) - f(t)\| < \delta$ for $t \in F_n \backslash E_n$, and $(g(t_n), h(t)) \le 1 - r$ for $t \in E_n$. Since

$$\begin{aligned}
1 - \frac{r\delta}{3} &\le \int_\Omega (\varphi(t), h(t))\, d\mu(t) \\
&= \int_{\Omega\backslash \bigcup_n E_n} (\varphi(t), h(t))\, d\mu(t) + \int_{\bigcup_n E_n} (\varphi(t), h(t))\, d\mu(t) \\
&\le \mu\left(\Omega \backslash \bigcup_n E_n\right) + \int_{\bigcup_n E_n} \left(\sum_n g(t_n)\chi_{F_n}, h(t)\right) d\mu(t) \\
&= \mu\left(\Omega \backslash \bigcup_n E_n\right) + \sum_n \int_{E_n} (g(t_n), h(t))\, d\mu(t) \\
&\le \mu\left(\Omega \backslash \bigcup_n E_n\right) + (1 - r)\mu\left(\bigcup_n E_n\right) = 1 - r\mu\left(\bigcup_n E_n\right),
\end{aligned}$$

we have $\mu(\bigcup_n E_n) \le \delta/3$. Thus

$$\begin{aligned}
\|h - f\|^p &= \int_{\Omega\backslash \bigcup_n F_n} \|h(t) - f(t)\|^p\, d\mu(t) + \int_{\bigcup_n F_n \backslash E_n} \|h(t) - f(t)\|^p\, d\mu(t) \\
&\quad + \int_{\bigcup_n E_n} \|h(t) - f(t)\|^p\, d\mu(t) \\
&\le 2^p \mu\left(\Omega \backslash \bigcup_n F_n\right) + \delta^p \mu\left(\bigcup_n F_n \backslash E_n\right) + 2^p \mu\left(\bigcup_n E_n\right) \\
&\le 2^p 2\delta/3 + \delta^p + 2^p\delta/3 = 2^p\delta + \delta^p < 3^p\delta. \quad \text{Q.E.D.}
\end{aligned}$$

If (Ω, Σ, μ) is a probability space and $f \in B_{L^p(\mu, X)}$ with $f(\Omega) \subset \text{str-exp}\, B_X$, then f must be a strongly exposed point of $B_{L^p(\mu, X)}$ as shown in Lemma 4.

Lemma 4. *Suppose* (Ω, Σ, μ) *is a probability space and* $f \in B_{L^p(\mu, X)}$ *with* $f(\Omega) \subset S_X$. *If there is a function* $g\colon \Omega \to S_{X^*}$ *such that, for every* $t \in \Omega$, $g(t)$ *strongly exposes* B_X *at* $f(t)$, *then there exist* $\varepsilon_n > 0$ *and* $\delta_n > 0$, *with* $\lim_n \varepsilon_n = 0$ *and* $\lim_n \delta_n = 0$, *and* $\{g_n\} \subset B_{L^q(\mu, X^*)}$ *satisfying*:

(1) $g_n(\Omega) \subset g(\Omega)$, *and* $\int_\Omega (g_n(t), f(t))\, d\mu(t) > 1 - \delta_n$, *and*

(2) *for* $n \ge k \ge 1$, *if* $h \in S(f)$ *and* $\int_\Omega (g_n(t), h(t))\, d\mu(t) > 1 - \delta_k$, *then* $\|f - h\| \le \varepsilon_k$.

Thus, by Lemma 2, we have $f \in \text{str-exp}\, B_{L^p(\mu, X)}$.

Proof. We may assume that $f(\Omega)$ is separable. For $k \ge 1$ and $m \ge 1$ set $\alpha_k = 2^{-k}$ and

$$D(m, k) = \{t\colon t \in \Omega \text{ and } \operatorname{diam} S(g(t), B_X, \tfrac{1}{m}) < \alpha_k\}.$$

It is obvious that $D(m, k) \subset D(m + 1, k)$. Since, for every $t \in \Omega$, $g(t)$ strongly exposes B_X at $f(t)$, we have $\Omega = \bigcup_m D(m, k)$ for each $k \ge 1$. Thus for every subset A of Ω, we have

$$(*) \qquad \mu^*(A) = \lim_m \mu^*(A \cap D(m, k)).$$

In particular, there is $m_1 \geq 1$ such that $\mu^*(D(m_1, 1)) > 1 - \alpha_1/3$. Choose a measurable set $E(1, 1)$ such that $E(1, 1) \supset D(m_1, 1)$ and $\mu(E(1, 1)) = \mu^*(D(m_1, 1))$. Let $E(1, 2) = \Omega \backslash E(1, 1)$. Then $\{E(1, 1), E(1, 2)\}$ is a partition of Ω.

Assume, for $1 \leq k \leq n$, we have chosen $m_1 < \cdots < m_n$ and partitions $\{E(k, 1), E(k, 2), \ldots, E(k, k), E(k, k+1)\}$ of Ω so that, for $1 \leq i \leq k$,

$$(**) \qquad \begin{aligned} &(1) \quad \mu(E(k, i)) = \mu^*(A(k, i)) \\ &\qquad \text{where } A(k, i) = E(k, i) \cap \left(\bigcap_{j=i}^{k} D(m_j, j) \right), \text{ and} \\ &(2) \quad \sum_{j=1}^{i} \mu(E(k, j)) > 1 - \frac{\alpha_i}{3}. \end{aligned}$$

Since $\sum_{j=1}^{i} \mu^*(A(n, j)) > 1 - \alpha_i/3$ and $\mu(E(n, n+1)) + \sum_{j=1}^{n} \mu^*(A(n, j)) = 1$, by $(*)$ there is $m_{n+1} > m_n$ such that

$$\sum_{j=1}^{i} \mu^*(A(n, j) \cap D(m_{n+1}, n+1)) > 1 - \frac{\alpha_i}{3} \quad \text{for } 1 \leq i \leq n$$

and

$$\mu^*(E(n, n+1) \cap D(m_{n+1}, n+1)) + \sum_{j=1}^{n} \mu^*(A(n, j) \cap D(m_{n+1}, n+1)) > 1 - \frac{\alpha_{n+1}}{3}.$$

Let $A(n+1, n+1) = E(n, n+1) \cap D(m_{n+1}, n+1)$, and let $A(n+1, j) = A(n, j) \cap D(m_{n+1}, n+1)$ for $1 \leq j \leq n$. Then, for $1 \leq j \leq n+1$, we can choose a measurable set $E(n+1, j)$ such that

$$A(n+1, j) \subset E(n+1, j) \subset E(n, j) \quad \text{and} \quad \mu(E(n+1, j)) = \mu^*(A(n+1, j)).$$

Let $E(n+1, n+2) = \Omega \backslash \bigcup_{j=1}^{n+1} E(n+1, j)$. Then $\{E(n+1, 1), \ldots, E(n+1, n+1), E(n+1, n+2)\}$ is a partition of Ω which satisfies $(**)$ for $1 \leq i \leq k = n+1$. By induction, there are natural numbers $m_1 < m_2 < \cdots$ and partitions $\{E(k, 1), \ldots, E(k, k+1)\}$ of Ω such that $(**)$ is true for $1 \leq i \leq k \leq n < \infty$.

Now fix $n \geq 1$. It is obvious that $f(A(n, k)) \subset \bigcup \{S(g(t), B_X, \alpha_n/3m_n) : t \in A(n, k)\}$ for $1 \leq k \leq n$. Since $\{S(g(t), B_X, \alpha_n/3m_n) : t \in A(n, k)\}$ is an open covering of $f(A(n, k))$ which is separable, there is a sequence $\{t_{nk}^j\}_{j \geq 1}$ in $A(n, k)$ such that

$$f(A(n, k)) \subset \bigcup_j S\left(g(t_{nk}^j), B_X, \frac{\alpha_n}{3m_n}\right).$$

Define

$$E(n, k, j) = E(n, k) \cap f^{-1} \left\{ S\left(g(t_{nk}^j), B_X, \frac{\alpha_n}{3m_n}\right) \Big\backslash \bigcup_{i<j} S\left(g(t_{nk}^i), B_X, \frac{\alpha_n}{3m_n}\right) \right\}.$$

Then $E(n, k, j)$ is measurable, and $E(n, k, j) \cap E(n, i, j') = \varnothing$ if $i \neq k$ or $j \neq j'$. Since $A(n, k) \subset \bigcup_j E(n, k, j) \subset E(n, k)$ and $\mu(E(n, k)) = \mu^*(A(n, k))$, we have

$$(***) \qquad \mu\left(\bigcup_{j\geq 1} E(n, k, j)\right) = \mu(E(n, k)).$$

Thus by $(**)$ we have $\mu(\bigcup\{E(n, k, j)\colon 1 \leq k \leq n \text{ and } j \geq 1\}) > 1 - \alpha_n/3$. By definition, for all t in $E(n, k, j)$ we have $(g(t^j_{nk}), f(t)) > 1 - \alpha_n/3m_n$. Hence,

$$\left(\sum_{1\leq k\leq n,\, j\geq 1} g(t^j_{nk})\chi_{E(n,k,j)}, f\right) > \left(1 - \frac{\alpha_n}{3m_n}\right)\mu\left(\bigcup\{E(n, k, j)\colon 1 \leq k \leq n \text{ and } j \geq 1\}\right).$$

Since, for each t in Ω, $(g(t), f(t)) = 1$, there is a measurable function $g_n\colon \Omega \to g(\Omega)$ such that

$$g_n|_{\bigcup_{1\leq k\leq n,\, j\geq 1} E(n,k,j)} = \sum_{1\leq k\leq n,\, j\geq 1} g(t^j_{nk})\chi_{E(n,k,j)}$$

and

$$(g_n, f) = \int_\Omega (g_n(t), f(t))\, d\mu(t) > 1 - \frac{\alpha_n}{3m_n}.$$

Let $\varepsilon_n = 3(\alpha_n)^{1/p}$ and $\delta_n = \alpha_n/3m_n$. To complete the proof it remains to verify the following claim.

Claim. For $n \geq k \geq 1$, if $h \in S(f)$ and $\int_\Omega (g_n(t), h(t))\, d\mu(t) > 1 - \delta_k$, then $\|f - h\| \leq \varepsilon_k$.

For $1 \leq i \leq k$ and $j \geq 1$, since $t^j_{ni} \in A(n, i) \subset D(m_k, k)$ (see $(**)(1)$), we have that

$$\operatorname{diam} S(g(t^j_{ni}), B_X, 1/m_k) < \alpha_k.$$

By $(**)$ and $(***)$, we have

$$\mu\left(\bigcup\{E(n, i, j)\colon 1 \leq i \leq k \text{ and } j \geq 1\}\right) = \sum_{i=1}^{k} \mu(E(n, i)) > 1 - \frac{\alpha_k}{3} > 1 - \frac{2\alpha_k}{3}.$$

It is obvious that

$$f(E(n, i, j)) \subset S(g(t^j_{ni}), B_X, \alpha_n/3m_n) \subset S(g(t^j_{ni}), B_X, 1/m_k)$$

and

$$g_n|_{\bigcup_{1\leq i\leq k,\, j\geq 1} E(n,i,j)} = \sum_{1\leq i\leq k,\, j\geq 1} g(t^j_{ni})\chi_{E(n,i,j)}.$$

By Lemma 3, we can conclude that, if $h \in S(f)$ and $\int_\Omega (g_n(t), h(t))\, d\mu(t) > 1 - \alpha_k/3m_k$, then $\|f - h\| \leq 3(\alpha_k)^{1/p}$; in other words, if $h \in S(f)$ and $\int_\Omega (g_n(t), h(t))\, d\mu(t) > 1 - \delta_k$, then $\|f - h\| \leq \varepsilon_k$. Thus the claim holds. Q.E.D.

The general case can be reduced to special case in Lemma 4 by using Lemmas 5 and 6.

Lemma 5. *Suppose (Ω, Σ, μ) is a positive finite measure space and $f \in S_{L^p(\mu, X)}$, and suppose there is $M > 0$ such that $1/M \le \|f(t)\| \le M$ for all t in Ω. Let $f_0(t) = f(t)/\|f(t)\|$ for $t \in \Omega$, and let $\mu_0 = \mu/\mu(\Omega)$. Then $f \in$ str-exp $B_{L^p(\mu, X)}$ if and only if $f_0 \in$ str-exp $B_{L^p(\mu_0, X)}$.*

Proof. Suppose $f_0 \in$ str-exp $B_{L^p(\mu_0, X)}$. Then there is $T \in S_{L^p(\mu_0, X)^*}$ which strongly exposes $B_{L^p(\mu_0, X)}$ at f_0. There is a function g from Ω to X^* such that g is weak* measurable (that is, for every x in X the real-valued function $g(\cdot)(x)$ is measurable), $\|g(\cdot)\| \in S_{L^q(\mu_0)}$, and for any $\varphi \in L^p(\mu, X)$ [EV, IT],

$$T(\varphi) = \int_\Omega (g(t), \varphi(t))\, d\mu(t).$$

By Theorem 2 [G2], we have $\|g(t)\| = 1$ and $g(t)$ strongly exposes B_X at $f_0(t)$ for almost all t in Ω. Let $h(t) = \|f(t)\|^{p-1} g(t)$ for all t in Ω. Then $\|h(\cdot)\|$ strongly exposes $B_{L^p(\mu)}$ at $\|f(\cdot)\|$, and $h(t)$ strongly exposes B_X at $f(t)/\|f(t)\|$ for almost all t in Ω. By Theorem 2 [G2] the functional in $L^p(\mu, X)^*$ represented by h strongly exposes $B_{L^p(\mu, X)}$ at f.

The proof of the converse is similar. Q.E.D.

Lemma 6 (see [S, Theorem, p. 154]). *Suppose $\{X_i\}_{i \in I}$ is a family of Banach spaces. Let $f = (f(i))_{i \in I} \in l^p(X_i)$. Then f is a strongly exposed point of the unit ball of $l^p(X_i)$ if and only if $\|f\| = 1$ and $f(i)/\|f(i)\| \in$ str-exp B_{X_i} for every $i \in \operatorname{supp} f$.*

Theorem 7. *Suppose f is an element in $L^p(\mu, X)$. If $\|f\| = 1$ and $f(t)/\|f(t)\| \in$ str-exp B_X for almost all $t \in \operatorname{supp} f$, then $f \in$ str-exp $B_{L^p(\mu, X)}$.*

Proof. Let $E_0 = f^{-1}(0)$, and, for each integer $n \ge 1$, let

$$E_{2n} = \{t \colon t \in \Omega,\ 2^{n-1} < \|f(t)\| \le 2^n\}$$

and

$$E_{2n-1} = \{t \colon t \in \Omega,\ 2^{-n} < \|f(t)\| \le 2^{-n+1}\}.$$

It is obvious that $\{E_n\}_{0 \le n < \infty}$ is a partition of Ω such that, for each n, $0 < n < \infty$, there is $M > 0$ with $1/M \le \|f(t)\| \le M$, for all t in E_n, and f is zero on E_0. Let μ_n be the restriction of μ to E_n, and let $X_n = L^p(\mu_n, X)$. Then the partition $\{E_n\}$ induces an isometry T from $L^p(\mu, X)$ onto $l^p(X_n)$, which is given by

$$T(h) = \{h|_{E_n}\}_{n \ge 1} \quad \text{for } h \text{ in } L^p(\mu, X).$$

By Lemmas 4 and 5, $f|_{E_n}/\|f|_{E_n}\| \in$ str-exp $B_{L^p(\mu_n, X)}$ whenever $f|_{E_n} \ne 0$. By Lemma 6 we have $f \in$ str-exp $B_{L^p(\mu, X)}$. Q.E.D.

References

[DU] J. Diestel and J. J. Uhl, Jr., *Vector measures*, Math. Surveys, vol. 15, Amer. Math. Soc., Providence, RI, 1977.

[EV] G. Emmanuele and A. Villani, *Lifting of rotundity properties from E to $L^p(\mu, E)$*, Rocky Mountain J. Math. **17** (1987), 617–627.

[G1] P. Greim, *Strongly exposed points in Bochner L^p-spaces*, Proc. Amer. Math. Soc. **88** (1983), 81–84.

[G2] ——, *A note on strong extreme and strongly exposed points in Bochner L^p-spaces*, Proc. Amer. Math. Soc. **93** (1985), 65–66.

[IT] A. Ionescu Tulcea and C. Ionescu Tulcea, *Topics in the theory of lifting*, Springer-Verlag, New York, 1969.

[J] J. A. Johnson, *Strongly exposed points in* $L^p(\mu, X)$, Rocky Mountain J. Math. **10** (1980), 517–519.

[S] M. Smith, *Rotundity and extremity in* $l^p(X_i)$ *and* $L^p(\mu, X)$, Contemp. Math., vol. 52, Amer. Math. Soc., Providence, RI, 1983, pp. 143–162.

Department of Mathematics and Statistics, Miami University, Oxford, Ohio 45056
E-mail address: zhu@miavx1.acs.muohio.edu

Department of Mathematics, University of Iowa, Iowa City, Iowa 52242
E-mail address: bllin@math.uiowa.edu