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COMPLEMENTED COPIES OF c_0 IN $L^\infty(\mu, E)$

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Abstract. Let E be a Banach space, and let (Ω, Σ, μ) be a measure space. We denote by $L^\infty(\mu, E)$ the Banach space of all E-valued μ-measurable essentially bounded functions on Ω, two functions being identified if they differ only on a locally μ-null set. We prove that if $L^\infty(\mu, E)$ contains a complemented copy of c_0, then E contains a copy of c_0.

Introduction

An important topic in the isomorphic theory of Banach spaces is to analyse when a Banach space contains a copy or a complemented copy of a certain classical Banach space such that c_0, l_∞, and so on. For the Banach spaces $L^p(\mu, E)$ of all vector-valued Bochner p-integrable functions ($1 \le p < +\infty$), this research has been done by Bourgain [1], Kwapien [2] (copies of c_0), Emmanuelle [5] (complemented copies of c_0), and Mendoza [9] (copies of l_∞). The aim of this paper is to study these problems for the Banach space $L^\infty(\mu, E)$. The notation and terminology used here can be found in [3, 4, 6].

Let (Ω, Σ, μ) be a measure space and E a Banach space. To avoid trivial cases, we assume that there is an infinite number of pairwise disjoint measurable sets of finite nonzero measure. A set $A \in \Sigma$ is said to be locally μ-null if $\mu(A \cap B) = 0$ for all $B \in \Sigma$ such that $\mu(B) < \infty$. A μ-measurable function f from Ω to E is said to be essentially bounded if for some real number $\varepsilon \ge 0$ the set $\{\omega \in \Omega : \|f(\omega)\| > \varepsilon\}$ is locally μ-null. Let $\operatorname{ess\,sup}(f)$ be the infimum of the set of all such numbers ε.

We denote by $L^\infty(\mu, E)$ the set of all essentially bounded μ-measurable functions from Ω to E, two functions identified if they differ only on a locally μ-null set. Under pointwise linear operations and the norm $\operatorname{ess\,sup}(\cdot)$, $L^\infty(\mu, E)$ is a Banach space.

Let $(\Delta_n)_{n\ge1}$ be a sequence of pairwise disjoint measurable sets of finite nonzero measure and x be a nonzero element of E. It is clear that the linear subspace H of $L^\infty(\mu, E)$ defined as

$$H = \left\{ \sum_{n=1}^{\infty} \alpha_n x \chi_{\Delta_n}(\cdot) : \ (\alpha_n) \in l_\infty \right\}$$

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is isomorphic to l_∞. Therefore, the only interesting question for $L^\infty(\mu, E)$ is about complemented copies of c_0 (l_∞ is always complemented [7, p. 133]).

One fact is trivial: if E contains a complemented copy of c_0, so does $L^\infty(\mu, E)$. On the other hand, Emmanuele [5] proved that if (Ω, Σ, μ) is not a purely atomic measure space and E contains a copy of c_0 (not necessarily complemented), then $L^p(\mu, E)$ ($1 \leq p < \infty$) contains a complemented copy of c_0. Looking carefully at his proof, one notices that the theorem is also true for $L^\infty(\mu, E)$. Therefore, to complete the study, we only have to answer the following question: Does E contain a copy of c_0, whenever $L^\infty(\mu, E)$ contains a complemented copy of c_0? Theorem 1 tell us that the answer is positive.

Main theorem

We start by recalling some facts on $L^\infty(\mu, E)$. Let us denote by $\mathscr{L}(\Omega, E)$ the linear space of all μ-measurable bounded functions from Ω to E. Note particularly that they are essentially bounded. Given $f \in \mathscr{L}(\Omega, E)$, we can consider its supremum norm

$$\|f\|_\infty = \sup\{\|f(\omega)\| : \omega \in \Omega\} < \infty.$$

Lemma 1 [6, p. 347]. *Given $F \in L^\infty(\mu, E)$, there exists $f \in \mathscr{L}(\Omega, E)$ such that $f \in F$. Moreover, for every $f \in F \cap \mathscr{L}(\Omega, E)$, the following holds:*

$$\operatorname{ess\,sup}(F) = \inf\{\|f\chi_{\Omega\setminus A}\|_\infty : A \text{ is a locally } \mu\text{-null set}\}.$$

Lemma 2. *Given $F \in L^\infty(\mu, E)$ and $f \in \mathscr{L}(\Omega, E)$ such that $f \in F$, there exists a locally μ-null set A such that for every locally μ-null set B including A we have* $\operatorname{ess\,sup}(F) = \|f\chi_{\Omega\setminus B}\|_\infty$.

Proof. By Lemma 1 and the characterization of the infimum, we can obtain a sequence $(A_n) \subset \Sigma$ of locally μ-null sets, such that

$$\|f\chi_{\Omega\setminus A_n}\|_\infty \leq \operatorname{ess\,sup}(F) + \frac{1}{n}.$$

Define $A = \bigcup_{n=1}^\infty A_n \in \Sigma$. Then, A is locally μ-null and for all $n \in \mathbb{N}$

$$\|f\chi_{\Omega\setminus A}\|_\infty \leq \|f\chi_{\Omega\setminus A_n}\|_\infty \leq \operatorname{ess\,sup}(F) + \frac{1}{n}.$$

Therefore, $\|f\chi_{\Omega\setminus A}\|_\infty = \operatorname{ess\,sup}(F)$. Finally, let B be a locally μ-null set including A. Then

$$\|f\chi_{\Omega\setminus B}\|_\infty \leq \|f\chi_{\Omega\setminus A}\|_\infty = \operatorname{ess\,sup}(F) \leq \|f\chi_{\Omega\setminus B}\|_\infty. \quad \square$$

Theorem 1. *If $L^\infty(\mu, E)$ contains a complemented copy of c_0, then E contains a copy of c_0.*

Proof. If $L^\infty(\mu, E)$ has a complemented copy of c_0, then there exists a sequence of (classes of) functions (F_n) in $L^\infty(\mu, E)$ and there exists a sequence (F_n') of continuous linear forms on $L^\infty(\mu, E)$ verifying the following two statements:

(1) There are real positive numbers γ_1, γ_2 such that

$$\gamma_1 \leq \operatorname{ess\,sup}\left(\sum_{n\in\sigma} F_n\right) \leq \gamma_2, \quad \text{for all finite subsets } \sigma \text{ of } \mathbb{N}.$$

(2) For each $F \in L^\infty(\mu, E)$, $(F_n'(F))$ is a null sequence and the map

$$T : L^\infty(\mu, E) \to c_0, \quad F \mapsto T(F) = (F_n'(F))$$

is linear and continuous. Moreover, $F_n'(F_m) = \delta_{mn}$ for all $n, m \in \mathbb{N}$, where δ_{mn} denotes as usual the Kronecker delta.

According to Lemma 1, there exists $f_n \in F_n \cap \mathscr{L}(\Omega, E)$ for every $n \in \mathbb{N}$. We notice that for every $\sigma \subset \mathbb{N}$ finite

$$\sum_{n\in\sigma} f_n \in \sum_{n\in\sigma} F_n.$$

Therefore, bearing in mind Lemma 2 and, for every $\sigma \subset \mathbb{N}$ finite, there exists $A_\sigma \in \Sigma$ locally μ-null such that

$$\operatorname{ess\,sup}\left(\sum_{n\in\sigma} F_n\right) = \left\|\left(\sum_{n\in\sigma} f_n\right)\chi_{\Omega\setminus A_\sigma}\right\|_\infty.$$

Let us define $A = \bigcup\{A_\sigma : \sigma \text{ is a finite subset of } \mathbb{N}\}$. Since the cardinal of the set of all finite subsets of natural numbers is numerable, we can assure that $A \in \Sigma$ is locally μ-null. Lemma 2 also implies that

$$\operatorname{ess\,sup}\left(\sum_{n\in\sigma} F_n\right) = \left\|\left(\sum_{n\in\sigma} f_n\right)\chi_{\Omega\setminus A}\right\|_\infty.$$

Therefore, for every $\omega \in \Omega \setminus A$ and for every $\sigma \subset \mathbb{N}$ finite, we have that

$$(*) \qquad \left\|\sum_{n\in\sigma} f_n(\omega)\right\| \le \left\|\left(\sum_{n\in\sigma} f_n\right)\chi_{\Omega\setminus A}\right\|_\infty = \operatorname{ess\,sup}\left(\sum_{n\in\sigma} F_n\right) \le \gamma_2.$$

In other words, the series $\sum_n f_n(\omega)$ is weakly unconditionally Cauchy in E. In fact, every subseries $\sum_k f_{n_k}(\omega)$ is weakly unconditionally Cauchy in E.

Claim. There are $\omega_0 \in \Omega \setminus A$ and a strictly increasing sequence $(n_k)_{k\ge1}$ of natural numbers such that the series $\sum_k f_{n_k}(\omega_0)$ does not converge in E.

Assuming the Claim, the theorem follows from a well-known result due to Bessaga and Pełczinsky [3, p. 45].

Proof of the Claim. Suppose that, for every $\omega \in \Omega \setminus A$, the series $\sum f_n(\omega)$ is subseries convergent in E. This allows the following construction. Let M be a subset (not necessarilly finite) of $\mathbb{N}$. We denote by $f(M)$ the following map from Ω to E:

$$\omega \mapsto f(M)(\omega) = \begin{cases} 0, & \omega \in A, \\ \sum_{n=1}^\infty \chi_M(n) f_n(\omega), & \omega \in \Omega \setminus A. \end{cases}$$

The reductio ad absurdum hypothesis assures that $f(M)$ is well defined.

Since f_n is μ-measurable, so is $f_n\chi_{\Omega\setminus A}$. We also note that the μ-measurable functions $\sum_{k=1}^n \chi_M(k) f_k \chi_{\Omega\setminus A}$ converge pointwise to $f(M)$; thus, $f(M)$ is μ-measurable.

On the other hand, taking limits in $(*)$, we deduce that

$$\sup\{\|f(M)(\omega)\| : \omega \in \Omega\} \le \gamma_2.$$

All these details mean that, for every $M \subset \mathbb{N}$, $f(M) \in \mathscr{L}(\Omega, E)$.

At this point, we need more definitions and notation. Given $f \in \mathscr{L}(\Omega, E)$, we denote its equivalence class in $L^\infty(\mu, E)$ by $[f]$. We also denote by m_0 the linear subspace of l_∞ formed by the sequences with a finite number of values. It is clear that every $\varphi = (\varphi(n)) \in m_0$ can be written in the form $\varphi(\cdot) = \sum_{i=1}^k \alpha_i \chi_{M_i}(\cdot)$, where the $\alpha_i \in \mathbb{K}$ ($i = 1, \ldots, k$) and the $M_i \subset \mathbb{N}$ are nonempty, pairwise disjoint and their union covers $\mathbb{N}$. The space m_0 endowed with the supremum norm is a barrelled space [3, p. 80].

Let us consider the following mapping

$$S : m_0 \to \mathscr{L}(\Omega, E), \quad \varphi = \sum_{i=1}^k \alpha_i \chi_{M_i} \mapsto S(\varphi) = \sum_{i=1}^k \alpha_i f(M_i).$$

Given $M = M_1 \cup M_2 \subset \mathbb{N}$ with $M_1 \cap M_2 = \varnothing$, we have $f(M) = f(M_1) + f(M_2)$. This shows that S is a well-defined linear map. Therefore, the map

$$[S] : m_0 \to L^\infty(\mu, E), \quad \varphi = \sum_{i=1}^k \alpha_i \chi_{M_i} \mapsto [S](\varphi) = \sum_{i=1}^k \alpha_i [f(M_i)]$$

is also well defined and linear.

Since m_0 is barrelled and $L^\infty(\mu, E)$ is a Banach space, the Closed Graph Theorem [7, p. 221] tell us that if $[S]$ has closed graph, then $[S]$ is continuous.

Thus, suppose that $(\varphi_p)_{p \in \mathbb{N}}$ is a null sequence in m_o and that the sequence $([S](\varphi_p))_p \subset L^\infty(\mu, E)$ converges to a certain $G \in L^\infty(\mu, E)$.

Take $g \in \mathscr{L}(\Omega, E) \cap G$. It is clear that

$$S(\varphi_p) - g \in \mathscr{L}(\Omega, E) \cap ([S](\varphi_p) - G).$$

By Lemma 2, for each $p \in \mathbb{N}$, we can obtain $C_p \in \Sigma$ locally μ-null such that

$$\|(S(\varphi_p) - g)\chi_{\Omega \setminus C_p}\|_\infty = \operatorname{ess\,sup}([S](\varphi_p) - G).$$

Let us set $C = \bigcup_{p=1}^\infty C_p \cup A \in \Sigma$. Since C is again locally μ-null and contains C_p, we have the fact that, for every $\omega \in \Omega \setminus C$,

$$\|S(\varphi_p)(\omega) - g(\omega)\| \leq \|(S(\varphi_p) - g)\chi_{\Omega \setminus C}\|_\infty = \operatorname{ess\,sup}([S](\varphi_p) - G) \overset{p \to \infty}{\longrightarrow} 0.$$

Therefore, for every $x' \in E'$ and for every $\omega \in \Omega \setminus C$, we deduce that

$$\lim_{p \to \infty} |\langle x', S(\varphi_p)(\omega)\rangle| = |\langle x', g(\omega)\rangle|.$$

On the other hand, since $\omega \notin A$, the series $\sum_n \langle x', f_n(\omega)\rangle$ is absolutely convergent. If we write $\sum_{n=1}^\infty |\langle x', f_n(\omega)\rangle| = K(x') < \infty$, then

$$|\langle x', S(\varphi_p)(\omega)\rangle| \leq \sum_{n=1}^\infty |\varphi_p(n)\langle x', f_n(\omega)\rangle| \leq K(x')\|\varphi_p\| \overset{p \to \infty}{\longrightarrow} 0.$$

The above limits must coincide, so we conclude that $\langle x', g(\omega)\rangle = 0$ for every $x' \in E'$; thus, $g(\omega) = 0$. That is to say, g is zero except in a locally μ-null set; then $G = 0$ and the mapping $[S]$ has closed graph.

Since m_0 is sequentially dense in l_∞, we can extend $[S]$ to a linear continuous map, which we denote $[S]^\infty$, from l_∞ to $L^\infty(\mu, E)$. In effect, given $\alpha \in l_\infty$, we can take a sequence $(\varphi_p)_{p \in \mathbb{N}}$ in m_0 which converges to α. It is clear that $([S](\varphi_p))$ is a Cauchy sequence in $L^\infty(\mu, E)$, so it converges to a certain $F \in L^\infty(\mu, E)$. We define $[S]^\infty(\alpha) = F$. It is easy to check that $[S]^\infty$ is a well-defined linear continuous map.

Let us consider the following composition of functions $\Pi = T \circ [S]^\infty$. Π is a linear continuous mapping from l_∞ to c_0. Moreover, for all $n \in \mathbb{N}$,

$$\Pi(e_n) = T([S]^\infty(e_n)) = T([f_n \chi_{\Omega \setminus A}]) = T(F_n) = (F'_m(F_n))_m = e_n$$

(e_n denotes the sequence which has values 1 in the n-place and vanishes in the other coordinates).

Since (e_n) is a basis for c_0, we have $\Pi(\alpha) = \alpha$ for every $\alpha \in c_0$. On the other hand, given $\alpha \in l_\infty$, there must exist $(\beta_n) \in c_0$ such that $\Pi(\alpha) = \sum_{n=1}^\infty \beta_n e_n$. From this, it follows that

$$\Pi^2(\alpha) = \Pi\left(\sum_{n=1}^\infty \beta_n e_n\right) = \sum_{n=1}^\infty \beta_n \Pi(e_n) = \sum_{n=1}^\infty \beta_n e_n = \Pi(\alpha).$$

Therefore, Π is a linear continuous projection from l_∞ onto c_0, which is known to be impossible [10]. □

Emmanuele's result leads one to think about the purely atomic case. In this line Leung and Räbiger [8] have studied the complemented copies of c_0 in the so-called l_∞-sum $\left(\sum_{i\in I} \bigoplus E_i\right)_\infty$ of an arbitrary family of Banach spaces $(E_i)_{i\in I}$. We note that if (Ω, Σ, μ) is a σ-finite purely atomic measure space, then there must be a countable number of atoms in Σ, so $L^\infty(\mu, E)$ can be identified with the Banach space $l_\infty(E) = (\sum_{n\in\mathbb{N}} \bigoplus E)_\infty$ of all bounded sequences in E. Combining our results with Leung, Räbiger, and Emmanuele results, we can set up the following dichotomy.

Corollary 1. *Let (Ω, Σ, μ) be a σ-finite measure space.*

(1) *If (Ω, Σ, μ) is purely atomic, then $L^\infty(\mu, E)$ contains a complemented copy of c_0 if and only if E contains a complemented copy of c_0.*
(2) *If (Ω, Σ, μ) is not purely atomic, $L^\infty(\mu, E)$ contains a complemented copy of c_0 if and only if E contains a copy of c_0.*

References

1. J. Bourgain, *An averaging result for c_0-sequences*, Bull. Soc. Math. Belg. Sér. B **30** (1978), 83–87.
2. S. Kwapien, *On Banach spaces containing c_0*, Studia Math. **52** (1974), 187–188.
3. J. Diestel, *Sequences and series in Banach spaces*, Springer-Verlag, New York, Berlin, Heidelberg, and Tokyo, 1984.
4. N. Dunford and J. T. Schwartz, *Linear operators, Part* I: *General theory*, Wiley-Interscience, New York, Chichester, and Brisbane, 1988.
5. G. Emmanuele, *On complemented copies of c_0 in L_X^p, $1 \le p < \infty$*, Proc. Amer. Math. Soc. **104** (1988), 785–786.
6. E. Hewitt and K. Stromberg, *Real and abstract analysis*, Springer-Verlag, New York, Heidelberg, and Berlin, 1965.
7. H. Jarchow, *Locally convex spaces*, Teubner, Stuttgart, 1981 .

8. D. Leung and F. Räbiger, *Complemented copies of* c_0 *in* l_∞*-sums of Banach spaces*, Illinois J. Math. **34** (1990), 52–58.

9. J. Mendoza, *Copies of* l_∞ *in* $L^p(\mu, X)$, Proc. Amer. Math. Soc. **109** (1990), 125–127.

10. R. J. Whitley, *Projecting* m *onto* c_0, Amer. Math. Monthly **73** (1966), 285–286.

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