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LINKS WITH UNLINKING NUMBER ONE ARE PRIME

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ABSTRACT. We prove that a link with unlinking number one is prime.

1. INTRODUCTION

Let L be a link in S^3. The *unlinking number of* L, $u(L)$, is defined to be the least number of times L must pass through itself in order to transform it into the unlink. It is not hard to see that this is the same as the minimum, over all diagrams of L, of the least number of crossing changes in that diagram needed to transform L to the unlink.

Recall that a *connected sum* of links L_1 and L_2 is any link L obtained by removing the interior of a trivial ball-pair $(B_i^3, B_i^1) \subset (S^3, L_i)$, $i = 1, 2$, and then gluing the resulting pairs along their boundaries (S_i^2, S_i^0). It is convenient to write $L = L_1 \# L_2$, although in general L is not uniquely determined by L_1 and L_2.

A natural conjecture, which is certainly classical at least in the case of knots, is the following.

Conjecture. $u(L_1 \# L_2) = u(L_1) + u(L_2)$.

We shall say that L is *prime* if $L = L_1 \# L_2$ implies that L_1 or L_2 is an unlink. (Note that although this is the definition of primality that is appropriate in our present context, it may not be the usual one; it differs from that given in [KT], for example.) Since $u(L) = 0$ if and only if L is an unlink, the above Conjecture implies that if $u(L) = 1$ then L is prime. In the case of knots, this was proved by Scharlemann [S]. Here we prove the analog for links with more than one component.

Theorem. *Let L be a link with more than one component such that $u(L) = 1$. Then L is prime.*

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An alternative proof of Scharlemann's theorem was given by Zhang [Z], by applying the main result of [GL1] to the 2-fold branched cover of the knot. The present proof follows the same philosophy, using [GL2] instead of [GL1].

2. Proof

Let K be a knot in the interior of an orientable 3-manifold M. Let $N(K)$ be a tubular neighborhood of K, and let α be the isotopy class of an essential simple closed curve in $\partial N(K)$. The manifold obtained by *α-Dehn surgery on K*, which we will denote by $K(\alpha)$, is the result of attaching a solid torus V to $M - \operatorname{int} N(K)$ by identifying ∂V with $\partial N(K)$ in such a way that α bounds a disk in V. If α and β are two such isotopy classes in $\partial N(K)$ then $\Delta(\alpha, \beta)$ denotes their minimal geometric intersection number.

If L is a link in S^3, let $M(L)$ denote the 2-fold branched cover of L, with branched covering projection $p : M(L) \to S^3$ and canonical involution $h : M(L) \to M(L)$.

Lemma 1. *Let L, L' be links in S^3 such that L' is obtained from L by a single crossing change. Then there exists a knot K in $M(L)$ such that*

(i) *K has an h-invariant tubular neighborhood $N(K)$ such that $p(N(K))$ is a 3-ball that meets L in an unknotted pair of arcs;*
(ii) *$M(L')$ is homeomorphic to $K(\alpha)$ for some α with $\Delta(\alpha, \mu) = 2$, where μ is the meridian of K.*

Proof. This follows from [L, Proof of Lemma 1]. (See also [M].) □

Note that if L is the n-component unlink, $n \geq 1$, then

$$M(L) \cong \mathop{\#}_{n-1} S^1 \times S^2.$$

The next lemma gives the converse.

Lemma 2. *Let L be a link in S^3 such that $M(L)$ is homeomorphic to $\#_{n-1} S^1 \times S^2$. Then L is the n-component unlink.*

Proof. The case $n = 1$ is the $\mathbb{Z}_2$-Smith Conjecture [W]. The general case follows easily by induction on n using [KT, Lemma 1]. □

Finally, we shall need the following fact about Dehn surgeries that yield reducible manifolds.

Lemma 3. *Let K be a knot in a 3-manifold M such that $M - K$ is irreducible but $K(\alpha)$ and $K(\beta)$ are reducible. Then*

$$\Delta(\alpha, \beta) \leq 1.$$

Proof. This is [GL2, Corollary 1.2]. □

Proof of Theorem. Let L be an n-component link in S^3, $n \geq 2$, with $u(L) = 1$. First note that if L is a split union $L_1 \sqcup L_2$, then clearly it must be that L_1 (say) is an unlink and $u(L_2) = 1$. Hence we may assume that L is nonsplit and, therefore, that $S^3 - L$ is irreducible.

Suppose for contradiction that L is a connected sum $L_1 \# L_2$, where L_i is not an unlink, $i = 1, 2$. Then $M(L) \cong M(L_1) \# M(L_2)$. In particular, since L_i is not the unknot, $i = 1, 2$, $M(L)$ is reducible, by the $\mathbb{Z}_2$-Smith Conjecture [W].

Since $u(L) = 1$, we can apply Lemma 1 with L' the n-component unlink, giving a knot K in $M(L)$, with meridian μ, such that

$$K(\alpha) \cong M(L') \cong \underset{n-1}{\#} S^1 \times S^2,$$

where $\Delta(\alpha, \mu) = 2$. But $K(\mu) \cong M(L)$ is also reducible. It follows from Lemma 3 that $X = M(L) - \operatorname{int} N(K)$ is reducible.

Choosing $N(K)$ as in part (i) of Lemma 1, let B_0 be the 3-ball $S^3 - \operatorname{int} p(N(K))$, and let $L_0 = L \cap B_0$. Then X is the 2-fold branched cover of (B_0, L_0). Since h restricts to an involution on X, [KT, Lemma 1] implies that X contains an essential 2-sphere S (one that does not bound a 3-ball) such that either $h(S) \cap S = \varnothing$ or $h(S) = S$ and S meets $p^{-1}(L)$ transversely.

In the first case, $p(S)$ is a 2-sphere in $B_0 - L_0$. Since $S^3 - L$ is irreducible by assumption, $p(S)$ bounds a 3-ball B in $B_0 - L_0$. But then B lifts to a 3-ball in X bounded by S, contradicting the essentiality of S.

In the second case, S must meet $p^{-1}(L)$ in two points. Then $p(S)$ is a 2-sphere in B_0 meeting L_0 in two points. Let B_1 be the 3-ball in B_0 bounded by $p(S)$, and let $L_1 = L \cap B_1$. Then in X, S bounds M_1, the 2-fold branched cover of (B_1, L_1). Since $M_1 \subset X \subset M(L') \cong \#_{n-1} S^1 \times S^2$, M_1 is homeomorphic to $\#_{k-1} S^1 \times S^2$ minus the interior of a 3-ball, where $k \leq n$. By Lemma 2 and the assumption that L is nonsplit, it follows that L_1 is an unknotted arc in B_1, giving $M_1 \cong B^3$ and again contradicting the essentiality of S. $\square$

Added in proof

After this paper was accepted for publication we learned that the theorem of the title is contained in Theorem 2 of [E-M]. So our paper should be regarded as giving a new proof of Eudave-Muñoz's result, in which the explicit combinatorial arguments of [E-M] are replaced by a reference to the main result of [GL2] (whose proof is based on more complicated combinatorial arguments). In the same way, our approach gives alternative proofs of Theorems 1, 2, and 3 of [E-M]. Eudave-Muñoz was also aware of the fact that the primality of unknotting number one knots follows from [GL1] (see [E-M, p. 775]).

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