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CHARACTERIZATIONS OF BOUNDED SETS IN SPACES OF ULTRADISTRIBUTIONS

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ABSTRACT. We characterize bounded sets in ultradistributions spaces $\mathscr{D}'^{(M_p)}_{L^t}$, $t \in [1, \infty]$, $S'^{\{M_p\}}$, and $S'^{(M_p)}$ and bounded sets and convergent sequences in $\mathscr{D}'^{(M_p)}$ and $\mathscr{D}'^{\{M_p\}}$ via the convolution by corresponding test functions. The structural theorems for $\mathscr{D}'^{\{M_p\}}_{L^t}$ and $\widetilde{\mathscr{D}}'^{\{M_p\}}_{L^t}$, $t \in [1, \infty]$, are also given.

1. INTRODUCTION

For the analysis of Beurling and Roumieau spaces of ultradistributions and the background information we refer to [6] and references therein. In this paper we investigate the bounded sets in ultradistribution spaces. The results are analogous to Schwartz's results for distributions.

A characterization of a bounded set in $\mathscr{D}'^{\{M_p\}}_{L^t}$, $t \in [1, \infty]$, and representation theorems for the elements of $\mathscr{D}'^{\{M_p\}}_{L^t}$, $t \in [1, \infty]$, are given in Lemma 2. In Lemma 3 spaces $\mathscr{D}^{\{M_p\}}_{L^t}$ and $\widetilde{\mathscr{D}}^{\{M_p\}}_{L^s}$, $s \in [1, \infty)$, and $\dot{\mathscr{B}}^{\{M_p\}}$ and $\widetilde{\dot{\mathscr{B}}}^{\{M_p\}}$ are compared. They are equal in the set-theoretical sense, but we do not know whether they are topologically equal. In Lemma 4 the projective limit representation of $\mathscr{S}^{\{M_p\}}$ is given. In these lemmas conditions (M.1) and (M.3)$'$ are assumed. In other assertions conditions (M.1), (M.2), and (M.3) are assumed. Theorems 1 and 2 are the so-called second structural theorems for bounded sets in spaces $\mathscr{D}'^{\{M_p\}}_{L^t}$, $t \in [1, \infty]$, and $\mathscr{S}'^*$ and for equicontinuous sets $\widetilde{\mathscr{D}}'^{\{M_p\}}_{L^t}$, $t \in [1, \infty]$. These theorems are based on the parameterixes for appropriate ultradifferential operators constructed in [6] (Lemma 5). In Theorem 3(i) it is proved that a set $B \subset \mathscr{D}'^{(M_p)}$ (resp. $B \subset \mathscr{D}'^{(M_p)}$) is bounded if for every $\phi \in \mathscr{D}'^{(M_p)}$ (resp. $\phi \in \mathscr{D}^{(M_p)}$) and every bounded open set Ω, $\sup\{|(f * \phi)(x)|; f \in B, x \in \Omega\} < \infty$. In Theorem 3(ii) an improvement of the quoted assertion is given, and in Theorem 4 the corresponding characterizations of a convergent sequence in $\mathscr{D}'^{(M_p)}$ (resp. $\mathscr{D}'^{\{M_p\}}$) are given.

2. NOTATION AND NOTIONS

The sets of real, complex, and natural numbers are denoted by $\mathbf{R}$, $\mathbf{C}$, $\mathbf{N}$;

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$\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n, \quad h^\alpha = h^{|\alpha|},$$

$$\alpha! = \alpha_1, \dots, \alpha!, \quad D_t = \frac{1}{i}\frac{d}{dt}, \quad D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}, \quad \partial^\alpha = \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

where $x, y \in \mathbf{R}^n$, $\alpha \in \mathbf{N}_0^n$, $h > 0$, $t \in \mathbf{R}$.

By L^s, $s \in [1, \infty]$, we denote the well-known space of functions f (classes) for which $|f|^s$ is integrable on $\mathbf{R}^n$. The norm in this space is denoted by $\| \|_{L^s}$; C^∞ is the space of smooth functions on $\mathbf{R}^n$. For an $f \in L^1$ the Fourier transform is defined by

$$\mathscr{F} f(\xi) = \widehat{f}(\xi) = \int_{\mathbf{R}^n} e^{-\langle x, \xi\rangle} f(x)\, dx, \qquad \xi \in \mathbf{R}^n.$$

By M_p, $p \in \mathbf{N}_0$, we denote a sequence of positive numbers. Assume $M_0 = 1$. The following conditions on these sequences will be considered.

(M.1) $M_p^2 \leq M_{p-1} M_{p+1}$, $p \in \mathbf{N}$;

(M.2)′ $M_p \leq A H^p M_{p-1}$, $p \in \mathbf{N}$, for some $A > 0$, $H > 0$;

(M.2) $M_p \leq A H^p M_q M_{p-q}$, $0 \leq q \leq p$, $p \in \mathbf{N}$, for some $A > 0$, $H > 0$;

(M.3)′ $\sum_{q=1}^{\infty} M_{q-1}/M_q < \infty$;

(M.3) $\sum_{q=p+1}^{\infty} M_{q-1}/M_q \leq A p M_p / M_{p+1}$, $p \in \mathbf{N}$, for some $A > 0$.

For the properties of such sequences we refer to [4] and [9]. In the definitions of test function spaces which are to follow we shall always assume that (M.1) and (M.3)′ hold.

Recall the definition of Beurling and Roumieau spaces of ultradifferentiable functions [4]. If K is a compact subset of $\mathbf{R}^n$, $h > 0$, and $\phi \in C^\infty$, then

$$\|\phi\|_{K,h} = \sup\left\{ \frac{|\partial^\alpha \phi(x)|}{h^\alpha M_{|\alpha|}};\ \alpha \in \mathbf{N}_0^n, x \in K \right\}.$$

Denote by $\mathscr{D}_{K,h}^{M_p}$ the space of functions ϕ from C^∞ for which $\operatorname{supp}\phi \subset K$ and $\|\phi\|_{K,h} < \infty$. The basic spaces of functions of classes (M_p) and $\{M_p\}$ are defined by

$$\mathscr{D}_K^{(M_p)} = \operatorname{proj}\lim_{h\to 0} \mathscr{D}_{K,h}^{M_p}, \qquad \mathscr{D}_K^{\{M_p\}} = \operatorname{ind}\lim_{h\to\infty} \mathscr{D}_{K,h}^{M_p},$$

$$\mathscr{D}^{(M_p)} = \operatorname{ind}\lim_{K\subset\subset\mathbf{R}^n} \mathscr{D}_K^{(M_p)}, \qquad \mathscr{D}^{\{M_p\}} = \operatorname{ind}\lim_{K\subset\subset\mathbf{R}^n} \mathscr{D}_K^{\{M_p\}}.$$

The notation $K \subset\subset \mathbf{R}^n$ means that K is compact and "grows" up to $\mathbf{R}^n$. For the properties of basic spaces which correspond to a sequence M_p which satisfies given conditions we refer to [4].

Denote by $\mathfrak{R}$ the set of all positive sequences r_p, $p \in \mathbf{N}$, which increase to ∞. This set is partially ordered and directed by the relation $r_p \preceq s_p$ defined by $r_p \leq s_p$, $p > p_0$, for some p_0.

Let $r_p \in \mathfrak{R}$ and K be a compact set in $\mathbf{R}^n$. Denote by $\mathscr{D}_{K,r_p}^{\{M_p\}}$ the space of smooth functions φ on $\mathbf{R}^n$ supported by K such that

$$\|\varphi\|_{K,r_p} = \sup\left\{ \frac{|\partial^\alpha \varphi(x)|}{M_{|\alpha|}(\prod_{i=1}^{|\alpha|} r_i)},\ \alpha \in \mathbf{N}_0^n,\ x \in K \right\} < \infty.$$

Clearly, this a Banach space. It is proved in [4] that under conditions (M.1), (M.2), and (M.3)

$$\mathscr{D}_K^{\{M_p\}} = \operatorname{proj}\lim_{r_p \in \Omega} \mathscr{D}_{K,r_p}^{\{M_p\}}.$$

Let Ω be a bounded open set in $\mathbf{R}^n$. We put

$$\mathscr{D}_{\Omega,r}^{(M_p)} = \operatorname{ind}\lim_{K \subset\subset \Omega} \mathscr{D}_{K,r}^{M_p}, \qquad \mathscr{D}_{\Omega,r_p}^{\{M_p\}} = \operatorname{ind}\lim_{K \subset\subset \Omega} \mathscr{D}_{K,r_p}^{\{M_p\}}.$$

These spaces will be used in Theorems 1 and 2.

Let $s \in [1, \infty]$. We define

$$\mathscr{D}_{L^s}^{(M_p)} = \operatorname{proj}\lim_{h \to 0} \mathscr{D}_{L^s,h}^{M_p}, \qquad \mathscr{D}_{L^s}^{\{M_p\}} = \operatorname{ind}\lim_{h \to \infty} \mathscr{D}_{L^s,h}^{M_p},$$

where $\mathscr{D}_{L^s,h}^{M_p}$ is the space of functions ϕ from C^∞ equipped with the norm

$$\|\phi\|_{L^s,h} = \sup\left\{\frac{\|\partial^\alpha \phi\|_{L^s}}{h^\alpha M_{|\alpha|}};\ \alpha \in \mathbf{N}_0^n\right\} < \infty.$$

The space $\mathscr{D}_{L^s}^{(M_p)}$ is defined in [10].

The common notation for the symbols (M_p) and $\{M_p\}$ will be $*$. If Ω is an open set in $\mathbf{R}^n$ and K is a compact subset of Ω, then $\mathscr{D}_K^*(\Omega)$ and $\mathscr{D}^*(\Omega)$ are defined analogously to the case $\Omega = \mathbf{R}$.

Since $\mathscr{D}^*$ is dense in $\mathscr{D}_{L^s}^*$, $s \in [1, \infty)$, and the inclusion mapping is continuous, it follows that the corresponding strong duals of $\mathscr{D}_{L^s}^*$ and $\mathscr{D}_{L^t}'^*$, $t = s/(s-1)$, are subspaces of Beurling and Roumieau ultradistribution spaces. We denote by $\dot{\mathscr{B}}^*$ the completion of $\mathscr{D}^*$ in $\mathscr{D}_{L^\infty}^*$. The strong dual of $\dot{\mathscr{B}}^*$ is denoted by $\mathscr{D}_{L^1}'^*$.

The space $\mathscr{S}^*$ is defined as follows [7, 8]. Denote by $S_h^{M_p}$, $h > 0$, the space of functions ϕ from C^∞ such that

$$\gamma_h(\phi) = \sup\left\{\frac{(1+|x|^2)^{\alpha/2}\|\partial^\beta \phi(x)\|_{L^2}}{h^{\alpha+\beta} M_{|\alpha|} M_{|\beta|}};\ \alpha, \beta \in \mathbf{N}_0^n\right\} < \infty.$$

The Hermite expansion of elements from $\mathscr{S}^*$ is determined in [8] under assumptions (M.1), (M.2), and (M.3)$'$, which imply that for $k > h$ the inclusion mapping $\mathscr{S}_h^{M_p} \to \mathscr{S}_k^{M_p}$ is nuclear. Thus,

$$S^{(M_p)} = \operatorname{proj}\lim_{h \to 0} S_h^{M_p} \qquad \left(S^{\{M_p\}} = \operatorname{ind}\lim_{h \to \infty} S_h^{M_p}\right)$$

is (FN) space ((LN) space) under conditions (M.1), (M.2), and (M.3)$'$.

Note that conditions (M.1) and (M.3)$'$ imply that the inclusion mapping $S_h^{M_p} \to S_k^{M_p}$, $k > h$, is compact [7].

The space $S^{\{p!^\alpha\}}$ is the Gelfand-Shilov-type space [2], and for $S^{(p!^\alpha)}$, $\alpha > 1$, we refer to [10]. $\mathscr{D}^*$ is dense in S^*, and, with the assumptions (M.1), (M.2), and (M.3)$'$, S^* is invariant under the Fourier transformation. So the same holds for the strong dual S'^*.

An operator of the form $P(D) = \sum_{\alpha \in \mathbf{N}_0^n} a_\alpha D^\alpha$, $a_\alpha \in \mathbf{C}$, is an (ultradifferential) operator of class (M_p) (resp. of class $\{M_p\}$) if there are constants $A > 0$, $h > 0$ (resp. for every $h > 0$ there is $A > 0$), such that

$$|a_\alpha| \leq A h^\alpha / M_{|\alpha|}, \qquad \alpha \in \mathbf{N}_0^n.$$

Special classes of entire functions [4] will be used. We recall some facts from [5]. In the sequel n' will be an integer greater than $n/2$ and $m_p = M_p/M_{p-1}$, $p \in \mathbf{N}$. Let $r > 0$. Put

(1)
$$P_r(\zeta) = (1+\zeta_1^2+\cdots+\zeta_n^2)^{n'} \prod_{i=1}^{\infty}\left(1+\frac{\zeta_1^2+\cdots+\zeta_n^2}{r^2 m_i^2}\right), \qquad \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbf{C}^n .$$

If (M.1), (M.2), and (M.3) hold, then $P_r(D)$ is an ultradifferentiable operator of class (M_p); it maps $\mathscr{D}^{(M_p)}$ into itself, and

(2)
$$\mathscr{F}(P_r(D)\phi)(\xi) = P_r(\xi)\hat{\phi}(\xi), \qquad \xi \in \mathbf{R}^n, \ \phi \in \mathscr{D}^{(M_p)}.$$

Put

(3)
$$P_{r_p}(\zeta) = (1+\zeta_1^2+\cdots+\zeta_1^2)^{n'} \prod_{i=1}^{\infty}\left(1+\frac{\zeta_1^2+\cdots+\zeta_n^2}{r_i^2 m_i^2}\right), \qquad \zeta \in \mathbf{C}^n .$$

Under conditions (M.1), (M.2), and (M.3) it is of class $\{M_p\}$. For elements of $\mathscr{D}^{\{M_p\}}$ and $P_{r_p}(D)$ equality (2) holds, as well.

The associated function for

$$N_p = M_p\left(\prod_{i=1}^{p} r_i\right), \qquad p \in \mathbf{N},$$

is defined by

$$N(\rho) = \sup\left\{\ln\frac{\rho^p 1}{N_p};\ p \in \mathbf{N}_0^n\right\}, \qquad \rho > 0.$$

If an element of $\mathfrak{R}$ is denoted by $\tilde{r}_p$, the corresponding associated function is denoted by $\widetilde{N}$. For the sequence M_p the associated function is denoted by M. From the definition it follows that for every $r_p \in \mathfrak{R}$, $C > 0$, and $c > 0$ there are $\tilde{r}_p \in \mathfrak{R}$ and $\rho_0 > 0$ such that

(4)
$$CN(c\rho) \le \widetilde{N}(\rho), \qquad \rho > \rho_0 .$$

Assume (M.1), (M.2), and (M.3). From [4, Proposition 4.5 and p. 91] it follows that there exist $D > 0$ and $c > 0$ such that

(5)
$$D\exp(-N(c|\xi|)) \le |1/P_{r_p}(\xi)| \le \exp(-N(\xi)), \qquad \xi \in \mathbf{R}^n .$$

By using the Cauchy formula

$$\partial^k(1/P_{r_p}(\xi)) = \frac{k!}{(2\pi i)^n}\int_{\Gamma_1}\cdots\int_{\Gamma_n}\frac{(1/P_{r_p})(\zeta)\,d\zeta_1\cdots d\zeta_n}{(\zeta_1-\xi_1)^{k_1+1}\cdots(\zeta_n-\xi_n)^{k_n+1}}, \qquad k \in \mathbf{N}_0^n,$$

where $\Gamma_i = \{\zeta_i | |\zeta_i - \xi_i| = d\}$, $i = 1, \dots, n$, $d > 0$, $\xi \in \mathbf{R}^n$, we obtain that there exists $C > 0$ such that

(6)
$$|\partial^k(1/P_{r_p}(\xi))| \le Ck!d^{-k}\exp(-N(\xi)/C), \qquad \xi \in \mathbf{R}^n .$$

The following two-dimensional version of [5, Lemma 3.4] is needed.

Lemma 1. *Let* $a_{p,q} > 0$, $p, q \in \mathbf{N}_0$.

(i) *There are* $h > 0$ *and* $C > 0$ *such that*

$$\sup\{a_{p,q}/h^{p+q};\ p, q \in \mathbf{N}_0\} \le C \tag{7}$$

if and only if for every sequence r_i, s_j *from* $\mathfrak{R}$

$$\sup\left\{\frac{a_{p,q}}{(\prod_{i=1}^{p} r_i \prod_{j=1}^{q} s_j)};\ p, q \in \mathbf{N}_0\right\} < \infty. \tag{8}$$

(ii) *There are sequences* r_i, $s_j \in \mathfrak{R}$ *and* $C > 0$ *such that*

$$\sup\left\{\left(\prod_{i=1}^{p} r_i \prod_{j=1}^{q} s_j\right) a_{p,q};\ p, q \in \mathbf{N}_0\right\} \le C$$

if and only if for every $h > 0$

$$\sup\{h^{p+q} a_{p,q};\ p, q \in \mathbf{N}_0\} < \infty.$$

Proof. One only has to prove the if parts.

(i) Assume that for every r_i, $s_j \in \mathfrak{R}$ (8) holds but for every $h > 0$ and $C > 0$ (7) does not hold. Let h_ν be a sequence which strictly increases to ∞. There exists a sequence (p_ν, q_ν) from $\mathbf{N}_0^2$ such that

$$p_{\nu+1} + q_{\nu+1} > p_\nu + q_\nu \quad \text{and} \quad h^{-(p_\nu+q_\nu)} a_{p_\nu, q_\nu} > \nu, \qquad \nu \in \mathbf{N}.$$

The following cases may appear:

(a) There is $p_0 \in \mathbf{N}_0$ such that (p_0, q_k) is a subsequence of (p_ν, q_ν) and q_k is strictly increasing.

(b) Symmetric case to previous one.

(c) There is a subsequence (p_k, q_k) of (p_ν, q_ν) such that both p_k and q_k are strictly increasing.

Let us prove that (c) implies the contradiction. The other two cases are similar. Let

$$\begin{gathered} r_i = h_1, \quad 1 \le i \le p_1, \qquad s_j = h_1, \quad 1 \le j \le q_1; \\ r_i = (h_k^{p_k} h_{k-1}^{-p_{k-1}})^{1/(p_k - p_{k-1})}, \qquad p_{k-1} < i \le p_k; \\ s_j = (h_k^{q_k} h_{k-1}^{-q_{k-1}})^{1/(q_k - q_{k-1})}, \qquad q_{k-1} < j < q_k, \ k = 2, 3, \ldots. \end{gathered}$$

The constructed sequences r_i and s_j do not satisfy (8), and this is a contradiction.

(ii) Put

$$\begin{aligned} b_\nu &= \sup\{a_{p,q};\ p + q = \nu,\ p, q \in \mathbf{N}_0\}, \qquad \nu \in \mathbf{N}_0; \\ C_h &= \sup\{h^\nu b_\nu;\ \nu \in \mathbf{N}_0\}, \qquad h \ge 1; \\ \widetilde{C}_h &= \sup\{h^{p+q} a_{p,q};\ p, q \in \mathbf{N}_0\}, \qquad h \ge 1, \\ H_\nu &= \sup\{h^\nu C_h^{-1};\ h \ge 1\}, \qquad \nu \in \mathbf{N}_0. \end{aligned}$$

Clearly $\widetilde{C}_h \le C_h$. Fix $p, q \in \mathbf{N}_0$, and let $\nu = p + q$. For every $h \ge 1$,

$$\sup\{h^\nu a_{p,q}/C_h;\ h \ge 1\} \le \sup\{\widetilde{C}_h/C_h;\ h \ge 1\} \le 1.$$

Thus, $\sup\{H_{p+q}a_{p,q};\ p, q \in \mathbf{N}_0\} \le 1$. Put $h_i = H_i/H_{i-1}$, $i \in \mathbf{N}$. The sequence h_i is increasing, and, for every $h > 0$, $H_p/h^p \to \infty$, $p \to \infty$. Since

$$\left(\prod_{i=1}^{p} h_i \prod_{j=1}^{q} h_j\right) a_{p,q} \le \prod_{i=1}^{p+q} h_i a_{p,q},$$

by taking $r_i = h_i$, $s_j = h_j$, we obtain

$$\left(\prod_{i=1}^{p} r_i \prod_{j=1}^{q} s_j\right) a_{p,q} < \infty,$$

and this implies the assertion.

3. Structural theorems

Denote by C_0 the space of continuous functions f on $\mathbf{R}^n$ such that $\lim_{|x|\to\infty} f(x) = 0$ equipped with the norm $\|\ \|_{L^\infty}$. Its dual space, the space of measures, is denoted by $\mathscr{M}^1$ (as in [3]), and we denote the dual norm in $\mathscr{M}^1$ by $\|\ \|_{\mathscr{M}^1}$. Note that, under conditions (M.1) and (M.3)$'$, $\mathscr{D}^*$ is dense in C_0.

Lemma 2. *Let M_p satisfy* (M.1) *and* (M.3)$'$.

(i) *A set $B \subset \mathscr{D}'^{\{M_p\}}_{L^t}$, $t \in (1, \infty]$, is bounded if and only if every $f \in B$ can be represented in the form*

$$f = \sum_{|\alpha|=0}^{\infty} D^\alpha f_\alpha, \qquad \text{where } f_\alpha \in L^t,\ \alpha \in \mathbf{N}^n,$$

such that for some $d > 0$ there exists $C > 0$ independent of $f \in B$ such that

$$\sum_{|\alpha|=0}^{\infty} d^\alpha M_{|\alpha|} \|f_\alpha\|_{L^t} < C.$$

(ii) *A set $B \subset \mathscr{D}'^{\{M_p\}}_{L^1}$ is bounded if and only if the representation of f in* (i) *holds with $f_\alpha \in \mathscr{M}^1$ and the condition in* (i) *holds with the norm $\|f_\alpha\|_{\mathscr{M}^1}$.*

(iii) *Let $f \in \mathscr{D}'^{\{M_p\}}$. It belongs to $\mathscr{D}'^{\{M_p\}}_{L^t}$, $t \in [1, \infty]$, if and only if f is of the form $f = \sum_{|\alpha|=0}^{\infty} D^\alpha f_\alpha$, where $f_\alpha \in L^t$ if $t \in (1, \infty]$ and $f_\alpha \in \mathscr{M}^1$ if $t = 1$, $\alpha \in \mathbf{N}_0^n$, such that for every $d > 0$*

$$\begin{cases} \sum_{|\alpha|=0}^{\infty} d^\alpha M_{|\alpha|} \|f_\alpha\|_{L^t} < \infty & \text{if } t \in (1, \infty], \\ \sum_{|\alpha|=0} d^\alpha M_{|\alpha|} \|f_\alpha\|_{\mathscr{M}^1} < \infty & \text{if } t = 1. \end{cases}$$

Proof. Clearly, the conditions given in (i)–(iii) are sufficient, so, we will prove that they are necessary.

(i) Since $\mathscr{D}^{(M_p)}_{L^s}$, $s = t/(t-1) \in [1, \infty)$, is barrelled, B is an equicontinuous set in $\mathscr{D}'^{(M_p)}_{L^t}$, and for some $d > 0$ and $C > 0$

$$|\langle f, \phi\rangle| \le C\|\phi\|_{L^s, d}, \qquad \phi \in \mathscr{D}^{(M_p)}_{L^s},\ f \in B.$$

Hence, by Hahn-Banach Theorem, B can be extended to an equicontinuous set B_1 on $\mathscr{D}^{M_p}_{L^s,d}$. Let $Y_{s,d}$ be the space of sequences (ϕ_α) from L^s such that

$$\|(\phi_\alpha)\|_{L^s,d} = \sup\left\{\frac{\|\phi_\alpha\|_{L^s}}{d^\alpha M_{|\alpha|}};\ \alpha \in \mathbf{N}_0^n\right\} < \infty$$

equipped with this norm. Again, by Hahn-Banach Theorem, B_1 can be extended to an equicontinuous set B_2 on $Y_{s,d}$. An equicontinuous set on $Y_{s,d}$ consists of sequences (f_α) from L^t for which (12) holds, and this implies assertion (i).

(ii) Let $X_{\infty,h}$ be the space of smooth functions ϕ such that, for every $\alpha \in \mathbf{N}_0^n$, $\phi^{(\alpha)} \in C_0$ and $\|\phi\|_{L^\alpha,h} < \infty$ equipped with the norm $\| \ \|_{L^\infty,h}$. There holds

$$\dot{\mathscr{B}}^{(M_p)} = \operatorname{proj}\lim_{h\to\infty} X_{\infty,h},$$

which implies that $\dot{\mathscr{B}}^{(M_p)}$ is barrelled. Thus in the same way as in (i) the proof of (ii) follows.

(iii) Let $Y_{s,h}$, $s \in [1,\infty]$, $h > 0$, be the space of sequences (ϕ_α), $\alpha \in \mathbf{N}_0^n$, from L^s and $s \in [1,\infty)$ and from C_0 for $s = \infty$ such that

$$\|(\phi_\alpha)\|_{L^s,h} = \sup\left\{\frac{\|\phi_\alpha\|_{L^s}}{h^\alpha M_{|\alpha|}};\ \alpha \in \mathbf{N}_0^n\right\} < \infty$$

equipped with the given norm.

Let $X_{s,h} = \mathscr{D}_{s,h}^{M_p}$, $s \in [1,\infty)$, $h > 0$, and $X_{\infty,h}$ be as in the proof of (ii). We identify $X_{s,h}$ with the corresponding subspace of $Y_{s,h}$, $s \in [1,\infty]$, $h > 0$, via the mapping $\phi \to (\phi^{(\alpha)})$. Note that $\dot{\mathscr{B}}^{\{M_p\}} = \operatorname{ind}\lim_{h\to\infty} X_{s,h}$. With the given identification we have

$$\mathscr{D}_{L^s}^{\{M_p\}} \subset Y_s = \operatorname{ind}\lim_{h\to\infty} Y_{s,h}, \qquad s \in [1,\infty),$$

$$\dot{\mathscr{B}}^{\{M_p\}} \subset Y_\infty = \operatorname{ind}\lim_{h\to\infty} Y_{\infty,h}.$$

Since the inclusion mappings are continuous, a continuous linear functional on $\mathscr{D}_{L^s}^{\{M_p\}}$ or $\dot{\mathscr{B}}^{\{M_p\}}$ is continuous on this space equipped with the induced topology from $\operatorname{ind}\lim_{h\to\infty} Y_{s,h}$, $s \in [1,\infty]$. Thus, Hahn-Banach Theorem implies assertion (iii) because in the set-theoretical sense we have

$$\left(\operatorname{ind}\lim_{h\to\infty} Y_{s,h}\right)' = \operatorname{proj}\lim_{h\to\infty} Y'_{s,h}, \qquad s \in [1,\infty].$$

Remark. With the notation as in (iii), for $s \in (1,\infty)$, the sequence $Y_{s,h}$, $h \in \mathbf{N}$, is weakly compact. This implies that $X_{s,h}$, $h \in \mathbf{N}$, and $Z_{s,h} = Y_{s,h}/X_{x,h}$, $h \in \mathbf{N}$, are weakly compact as well, and thus the dual Mittag-Leffler Lemma [4, Lemma 1.4] implies that the sequence

$$0 \leftarrow \operatorname{proj}\lim_{h\to\infty} X'_{s,h} \leftarrow \operatorname{proj}\lim_{h\to\infty} Y'_{s,h}$$

is exact, where (in the topological sense)

$$\operatorname{proj}\lim_{h\to\infty} X'_{s,h} = X'_s = \left(\operatorname{ind}\lim_{h\to\infty} X_{s,h}\right)',$$

$$\operatorname{proj}\lim_{h\to\infty} Y'_{s,h} = Y'_s = \left(\operatorname{ind}\lim_{h\to\infty} Y_{s,h}\right)'.$$

This implies that $\mathscr{D}_{L^s}^{\{M_p\}}$ and X_s equipped with the induced topology from Y_s have the same strong duals [4, Lemma 1.4(iii)]. We do not know whether the space X_s with the induced topology is quasi-barrelled, and because of that we do not have the characterization of a bounded set in $\mathscr{D}'^{\{M_p\}}_{L^t}$, $t \in (1, \infty)$.

Denote by $\mathscr{D}^{\{M_p\}}_{L^s, r_p}$, $r_p \in \mathfrak{R}$, $s \in [1, \infty]$, the space of smooth functions φ such that

$$\|\varphi\|_{L^s, r_p} = \sup\left\{ \frac{\|\partial^\alpha \varphi\|_{L^s}}{M_{|\alpha|}(\prod_{i=1}^{|\alpha|} r_i)};\ \alpha \in \mathbf{N}_0^n \right\} < \infty$$

equipped with this norm, and let $\widetilde{\mathscr{D}}^{\{M_p\}}_{L^s} = \operatorname{proj}\lim_{r_p \in \mathfrak{R}} \mathscr{D}^{\{M_p\}}_{L^s, r_p}$. The completion of $\mathscr{D}^{\{M_p\}}$ with respect to $\widetilde{\mathscr{D}}^{\{M_p\}}_{L^\infty}$ is denoted by $\dot{\widetilde{\mathscr{B}}}^{\{M_p\}}$. The corresponding dual spaces are denoted by $\widetilde{\mathscr{D}}'^{\{M_p\}}_{L^t}$, $t = s/(s-1) \in (1, \infty]$, and $\widetilde{\mathscr{D}}'^{\{M_p\}}_{L^1}$.

Lemma 3. *Let M_p satisfy* (M.1) *and* (M.3)$'$.

(i) $\widetilde{\mathscr{D}}'^{\{M_p\}}_{L^s} = \mathscr{D}^{\{M_p\}}_{L^s}$, $s \in (1, \infty)$, *in the set-theoretical sense. The same holds for $\dot{\mathscr{B}}^{\{M_p\}}$ and $\dot{\widetilde{\mathscr{B}}}^{\{M_p\}}$.*

(ii) *The inclusion mappings $i: \mathscr{D}^{\{M_p\}}_{L^s} \to \widetilde{\mathscr{D}}^{\{M_p\}}_{L^s}$, $s \in [1, \infty)$, and $i: \dot{\mathscr{B}}^{\{M_p\}} \to \dot{\widetilde{\mathscr{B}}}^{\{M_p\}}$ are continuous.*

Proof. Note that $\dot{\widetilde{\mathscr{B}}}^{\{M_p\}} = \operatorname{proj}\lim_{r_p \in \mathfrak{R}} X_{\infty, r_p}$, where X_{∞, r_p} is the space of smooth functions ϕ such that

$$\phi^{(\alpha)} \in C_0, \quad \alpha \in \mathbf{N}_0^n, \qquad \|\phi\|_{L^\infty, r_p} < \infty$$

equipped with this norm.

The proof of (i) follows from [5, Lemma 3.4], and (ii) follows from the inequality

$$\|\phi\|_{L^s, r_p} \le C_{r_p, h} \|\phi\|_{L^s, h}, \qquad \phi \in \mathscr{D}^{\{M_p\}}_{L^s, h},$$

where $r_p \in \mathfrak{R}$, $h > 0$, and $C_{r_p, h}$ is a suitable constant.

Lemma 4. *Let M_p satisfy* (M.1) *and* (M.3)$'$.

(i) *A set $B \subset \mathscr{S}'^{(M_p)}$, resp. $\mathscr{S}'^{\{M_p\}}$, is bounded if and only if every $f \in B$ can be represented in the form*

$$f = \sum_{\substack{|\alpha|=0 \\ |\beta|=0}}^{\infty} D^\alpha((1+|x|^2)^{\beta/2} f_{\alpha, \beta}), \qquad \textit{where } f_{\alpha, \beta} \in L^2,\ \alpha, \beta \in \mathbf{N}_0^n,$$

such that for some $d > 0$ (resp. every $d > 0$) there exists $D > 0$ independent of $f \in B$ such that

$$\sum_{\substack{|\alpha|=0 \\ |\beta|=0}}^{\infty} d^{\alpha+\beta} M_{|\alpha|} M_{|\beta|} \|f_{\alpha, \beta}\|_{L^2} < D.$$

(ii) $\mathscr{S}^{\{M_p\}} = \operatorname{proj}\lim_{r_i, s_j \in \mathfrak{R}} \mathscr{S}^{M_p}_{r_i, s_j}$ *where $\mathscr{S}^{M_p}_{r_i, s_j}$ is the space of functions φ from C^∞ such that*

$$\gamma_{r_i, s_j}(\varphi) = \sup\left\{ \frac{\|(1+|x|^2)^{\beta/2} \partial^\alpha \varphi\|_{L^2}}{M_{|\alpha|}(\prod_{i=1}^{|\alpha|} r_i) m_{|q|}(\prod_{j=1}^{|\beta|} s_j)};\ \alpha, \beta \in \mathbf{N}_0^n \right\} < \infty.$$

Proof. (i) We shall prove the assertion in the $\{M_p\}$ case because it is rather complicated. Note that $\mathcal{S}^{\{M_p\}}$ is barrelled, and thus B is an equicontinuous subset of $\mathcal{S}'^{\{M_p\}}$.

Let W_h, $h > 0$, be the space of sequences $(\phi_{\alpha,\beta})$, $\alpha, \beta \in \mathbf{N}_0^n$, from L^2 such that

$$\|(\phi_{\alpha,\beta})\|_{L^s,h} = \sup\left\{\frac{\|\phi_{\alpha,\beta}\|_{L^2}}{h^{\alpha+\beta}M_{|\alpha|}M_{|\beta|}};\ \alpha,\ \beta \in \mathbf{N}_0^n\right\} < \infty$$

equipped with this norm. We identify $\mathcal{S}_h^{M_p}$ by the corresponding subspace of W_h. Since W_h, $h \in \mathbf{N}$, is a weakly compact inductive sequence and $\mathcal{S}_h^{M_p}$, $h \in \mathbf{N}$, is a compact one, [4, Lemma 1.4(iii)] implies that the sequence

$$0 \leftarrow \operatorname{proj}\lim_{h\to\infty} (\mathcal{S}_h^{M_p})' \leftarrow \operatorname{proj}\lim_{h\to\infty} W_h'$$

is exact, where

$$\operatorname{proj}\lim_{h\to\infty} (\mathcal{S}_h^{M_p})' = \mathcal{S}'^{\{M_p\}} = \left(\operatorname{ind}\lim_{h\to\infty} \mathcal{S}_h^{M_p}\right)',$$

$$\operatorname{proj}\lim_{h\to\infty} W_h' = W' = \left(\operatorname{ind}\lim_{h\to\infty} W_h\right)'.$$

Since $\mathcal{S}^{\{M_p\}}$ is a Montel space by [4, Lemma 1.4(v)], $\mathcal{S}^{\{M_p\}}$ is a closed subspace of W, and by Hahn-Banach Theorem the equicontinuous set $B \subset \mathcal{S}'^{\{M_p\}}$ can be extended to the equicontinuous set $\widetilde{\mathcal{B}}$ in W'. Thus $\widetilde{B}$ consists of the sequence $(f_{\alpha,\beta})$, α, $\beta \in \mathbf{N}_0^n$, from L^2 such that for every $d \in \mathbf{N}$ there is $C > 0$ which is the same for all the elements from $\widetilde{B}$ such that

$$\sum_{\substack{|\alpha|=0\\|\beta|=0}}^{\infty} d^{\alpha+\beta} M_{|\alpha|}M_{|\beta|}\|f_{\alpha,\beta}\|_{L^2} < C.$$

The mapping

$$\sum_{\substack{|\alpha|=0\\|\beta|=0}}^{\infty} (-1)^{\alpha} D^{\alpha}(1+|x|^2)^{\beta/2}$$

maps W' onto $\mathcal{S}'^{\{M_p\}}$ and

$$B \subset \left(\sum_{\substack{|\alpha|=0\\|\beta|=0}}^{\infty} (-1)^{\alpha} D^{\alpha}(1+|x|)^{\beta/2}\right)\widetilde{B},$$

which implies the assertion.

(ii) From Lemma 1 it follows that $\varphi \in C^\infty(\mathbf{R}^n)$ belongs to $\mathcal{S}^{\{M_p\}}$ if and only if $\gamma_{r_i,s_j}(\phi) < \infty$ for every r_i, $s_j \in \mathfrak{R}$.

Every norm γ_{r_i,s_j}, $r_i, s_j \in \mathfrak{R}$, is continuous on $\mathcal{S}_h^{M_p}$, $h > 0$, so it is continuous on $\mathcal{S}^{\{M_p\}}$.

Since $\mathcal{S}^{\{M_p\}}$ is reflexive, every continuous seminorm p is bounded by the seminorm p^B, where B is bounded in $\mathcal{S}'^{\{M_p\}}$, defined by

$$p^B(\varphi) = \sup\{|\langle f, \varphi\rangle|;\ f \in B\}.$$

From it follows

$$p^B(\varphi) \leq \sup_{f\in B} \sum_{\substack{|\alpha|=0 \\ |\beta|=0}} \|(1+|x|^2)^{\beta/2} D^\alpha \varphi\|_{L^2} \|f_{\alpha,\beta}\|_{L^2},$$

and by Lemma 1 it follows that there exist r_i and s_j from $\mathfrak{R}$ such that for some $C > 0$

$$p^B(\varphi) \leq C\gamma_{r_i, s_j}(\phi).$$

The proof is completed.

Up to the end of the paper we shall assume that conditions (M.1), (M.2), and (M.3) hold.

The following assertion of Komatsu will be used. Note that the first part of this assertion is also proved in [1].

Lemma 5 [6]. *Let K be a compact neighbourhood of zero, $r > 0$, and $r_p \in \mathfrak{R}$.*

(i) *There is $u \in \mathcal{D}^{(M_p)}_{K, r/2}$ and $\psi \in \mathcal{D}^{(M_p)}_K$ such that*

$$P_r(D)u = \delta + \psi, \tag{9}$$

where P_r is of form (1).

(ii) *There are $u \in C^\infty$ and $\psi \in \mathcal{D}^{\{M_p\}}_K$ such that*

$$\begin{gathered} P_{r_p}(D)u = \delta + \psi, \\ \operatorname{supp} u \subset K, \qquad \sup_{x\in K}\left\{\frac{|\partial^\alpha u(x)|}{R_{|\alpha|}M_{|\alpha|}}\right\} \to 0, \qquad |\alpha| \to \infty, \end{gathered} \tag{10}$$

where P_{r_p} is of the form (3).

Theorem 1. *Let $A \subset \mathcal{D}'^*$.*

(i) *A is a bounded subset of $\mathcal{D}'^{\{M_p\}}_{L^t}$, $t \in [1, \infty]$, if and only if there are $r > 0$ and bounded sets A_1 and A_2 in L^t such that every $f \in A$ is of the form*

$$f = P_r(D)F_1 + F_2, \qquad F_1 \in A_1, \ F_2 \in A_2.$$

(ii) *A is an equicontinuous subset of $\widetilde{\mathcal{D}}'^{\{M_p\}}_{L^t}$, $t \in [1, \infty]$, if and only if there are $r_p \in \mathfrak{R}$ and bounded sets A_1 and A_2 in L^t such that every $f \in A$ is of the form*

$$f = P_{r_p}(D)F_1 + F_2, \qquad F_1 \in A_1, \ F_2 \in A_2. \tag{11}$$

Proof. Note that we do not know whether the basic space is quasi-barrelled, and because of that we assume in (ii) that A is equicontinuous. We shall prove only assertion (ii) because it is rather complicated. Note that (i) is proved in [1] for $A = \{f\}$.

Since P_{r_p} maps $\mathcal{D}^{\{M_p\}}_{L^s}$, $s \in [1, \infty)$, and $\dot{\mathcal{B}}^{\{M_p\}}$ continuously into the same spaces, (11) implies that A is bounded in $\mathcal{D}'^{\{M_p\}}_{L^t}$.

We shall prove the converse assertion for

$$t = s/(s-1), \qquad s \geq 1.$$

For $t = 1$ $(s = \infty)$ the proof is similar.

Let Ω be a bounded open set in $\mathbf{R}^n$ which contains zero, $K = \bar{\Omega}$, and $\phi \in \mathscr{D}_K^{\{M_p\}}$. First we show that, for every $f \in A$, $f * \phi$ is a continuous linear functional on $\mathscr{D}^{\{M_p\}}$ endowed with the topology of L^s. Since A is equicontinuous, there are $C > 0$ which do not depend on $f \in A$ and $r_p \in \mathfrak{R}$ such that for every $\psi \in \mathscr{D}^{\{M_p\}}$

$$(12) \qquad \begin{aligned} |\langle f * \phi, \psi\rangle| = |\langle f, \check{\phi} * \psi\rangle| &\leq C\|\check{\phi} * \psi\|_{L^s, r_p} \\ &\leq C\|\phi\|_{K, r_p}\|\psi\|_{L^s} \leq C_1\|\psi\|_{L^s} \qquad (\check{\phi}(-x) = \phi(x)). \end{aligned}$$

Because $\mathscr{D}^{\{M_p\}}$ is dense in L^s, it follows that $\{f * \phi;\ f \in A\}$ is a set of (continuous) functions bounded in L^t. Moreover, (12) implies

$$\sup\{\|f * \phi\|_{L^t};\ f \in A\} \leq C\|\phi\|_{K, r_p}.$$

So if B is a bounded set of $\mathscr{D}_K^{\{M_p\}}$, then

$$\sup\{\|f * \phi\|_{L^t};\ \phi \in B,\ f \in A\} < \infty.$$

Next, we show that there is (another) $r_p \in \mathfrak{R}$ such that for every $\theta \in \mathscr{D}_{\Omega, r_p}^{\{M_p\}}$, $\{f * \theta;\ f \in A\}$ is a bounded set in L^t.

Let B_1 be the unit ball in L^s and B be a bounded subset of $\mathscr{D}_K^{\{M_p\}}$. Then for every $f \in A$, $\psi \in B_1 \cap \mathscr{D}^{\{M_p\}}$, and $\phi \in B$,

$$|\langle f * \check{\psi}, \check{\phi}\rangle| = |\langle f * \phi, \psi\rangle| \leq \|f * \phi\|_{L^t}\|\psi\|_{L^s} = \|f * \phi\|_{L^t} \leq D < \infty,$$

where D does not depend on ϕ and f. This implies that $\{f * \check{\psi};\ f \in A,\ \psi \in B_1 \cap \mathscr{D}^{\{M_p\}}\}$ is bounded in $\mathscr{D}_K^{\prime\{M_p\}}$. Since $\mathscr{D}_K^{\{M_p\}}$ is barrelled, this family is equicontinuous in $\mathscr{D}_K^{\prime\{M_p\}}$. This implies that there exists a neighbourhood of zero in $\mathscr{D}_K^{\{M_p\}}$

$$V_{r_p}(\varepsilon) = \{\theta \in \mathscr{D}_K^{\{M_p\}};\ \|\theta\|_{K, r_p} \leq \varepsilon\}, \qquad \varepsilon > 0,$$

such that

$$\theta \in V_{r_p}(\varepsilon) \Rightarrow |\langle f * \check{\psi}, \check{\theta}\rangle| = |\langle f * \theta, \psi\rangle| \leq 1, \qquad \psi \in B_1 \cap \mathscr{D}^{\{M_p\}},\ f \in A.$$

The same inequality holds for the closure of $V_{r_p}(\varepsilon)$ in $\mathscr{D}_{K, r_p}^{\{M_p\}}$.

Let $\delta_k(t) = k\omega(kt)$, $t \in \mathbf{R}$, $k \in \mathbf{N}$, where $\omega \in \mathscr{D}^{\{M_p\}}$, $0 \leq \omega \leq 1$, $\int_{\mathbf{R}} \omega(t)\,dt = 1$, and let $\delta_k(x) = \delta_k(x_1), \dots, \delta_k(x_n)$, $x \in \mathbf{R}^n$, $k \in \mathbf{N}$. One can easily prove that for given $\mu \in \mathscr{D}_{\Omega, r_p}^{\{M_p\}}$, $\mu * \delta_k$, $k \in \mathbf{N}$, $k > k_0$, is a sequence from $\mathscr{D}_K^{\{M_p\}}$ which converges to μ in the norm $\|\ \|_{K, r_p}^{\{M_p\}}$. If $\theta \in \mathscr{D}_{\Omega, r_p}^{\{M_p\}}$, then, for some $N > 0$, $\|\theta/N\|_{K, r_p}^{\{M_p\}} < \varepsilon$ and there is a sequence from $\overline{V_{r_p}(\varepsilon)}$ which converges to θ/N in the norm $\|\ \|_{K, r_p}^{\{M_p\}}$.

This implies that for every $\theta \in \mathscr{D}_{\Omega, r_p}^{\{M_p\}}$ there is $C > 0$ such that

$$|\langle f * \check{\psi}, \check{\theta}\rangle| = |\langle f * \theta, \psi\rangle| \leq C, \qquad \psi \in B_1 \cap \mathscr{D}^{\{M_p\}},\ f \in A,$$

and thus,

$$|\langle f*\theta\,,\,\psi\rangle| \le C\|\psi\|_{L^s}\,, \qquad \psi\in\mathscr{D}^{\{M_p\}}\,,\ f\in A\,.$$

This proves that, for every $\theta\in\mathscr{D}^{\{M_p\}}_{\Omega,r_p}$, $\{f*\theta;\ f\in A\}$ is a bounded set in L^t. Lemma 5(ii) implies that for every $f\in A$

$$f = P_{r_p}(u*f) - \psi * f\,, \qquad \text{where } u\in\mathscr{D}^{\{M_p\}}_{\Omega,r_p}\,,\ \psi\in\mathscr{D}^{\{M_p\}}_{\Omega}\,,$$

and since $\{u*f;\ f\in A\}$ and $\{\psi*f;\ f\in A\}$ are bounded sets in L^t, the proof is completed.

Let $r>0$ (resp. $r_p\in\mathfrak{R}$) be given. There is $\tilde r>0$ (resp. $\tilde r_p\in\mathfrak{R}$) such that for $\varphi\in\mathscr{D}^{(M_p)}_{K,\tilde r/2}$ (resp. $\varphi\in\mathscr{D}^{\{M_p\}}_{K,\tilde r_p}$), $P_r\varphi$ (resp. $P_{r_p}\varphi$) is a continuous function. This and the preceding theorem imply the following

Corollary 1. *An $f\in\mathscr{D}'^*$ is from $\mathscr{D}'^{(M_p)}_{L^t}$ (resp. $\widetilde{\mathscr{D}}'^{\{M_p\}}$), $t\in[1,\infty]$, if and only if for every compact set K there is $r>0$ (resp. $r_p\in\mathfrak{R}$) such that for every $\phi\in\mathscr{D}^{(M_p)}_{K,r/2}$ (resp. $\phi\in\mathscr{D}^{\{M_p\}}_{K,r_p}$) $f*\phi\in L^t$.*

The following is the structural theorem for tempered ultradistributions.

Theorem 2. *Let $A\subset\mathscr{D}'^{(M_p)}$ (resp. $A\subset\mathscr{D}'^{\{M_p\}}$). Then A is a bounded subset of $S'^{(M_p)}$ (resp. A is a bounded subset of $S'^{\{M_p\}}$) if and only if f is of the form*

$$f = P(D)F\,, \qquad F\in A_1\,, \tag{13}$$

where P is an operator of class (M_p) (resp. of class $\{M_p\}$) and A_1 is a set of continuous function on $\mathbf{R}^n$ such that for some $k>0$ and some $C>0$ (resp. for every $k>0$ there is $C>0$) such that

$$|F(x)| \le C\exp(M(k|x|))\,, \qquad F\in A_1\,,\ x\in\mathbf{R}^n\,. \tag{14}$$

Proof. We shall prove again the $\{M_p\}$ case since this is rather complicated and the ideas of the proofs for both cases are similar. Clearly (14) implies that by (13) a bounded set in $\mathscr{S}'^{\{M_p\}}$ is defined.

Let A be a bounded set in $S'^{\{M_p\}}$. For the Fourier transform $\hat f$ $(f\in A)$ there are r_i, $s_j\in\mathfrak{R}$ and $A>0$ which do not depend on $f\in A$ such that

$$|\langle \hat f(\xi)\,,\,\phi(\xi)\rangle| > A\gamma_{r_i,s_j}(\phi)\,, \qquad \phi\in S^{\{M_p\}}\,. \tag{15}$$

For some $D>0$ and $c>0$.

$$\sup_{\alpha\in N_0^n}\left\{\frac{(1+|x|^2)^{\alpha/2}}{M_{|\alpha|}(\prod_{i=1}^{|\alpha|} r_i)}\right\} \le D\,\exp(N(c|x|))\,, \qquad x\in\mathbf{R}^n\,. \tag{16}$$

Let $\tilde r_p$ and ρ_0 correspond to r_p, c and C in (4) where C is from (6) and c

from (16). If $\phi \in \mathscr{D}^{\{M_p\}}$, from (4)–(6), it follows that

$$\begin{aligned}
&\gamma_{r_i, s_j}(\phi/P_{\tilde{r}_p}) \\
&\quad \le \sup\left\{ \frac{\|(1+|x|^2)^{\alpha/2} \sum_{k\le\beta} \binom{\beta}{k} \partial^{\beta-k}\phi \partial^k(1/P_{\tilde{r}_p})\|_{L^2}}{M_{|\alpha|}(\prod_{i=1}^{|\alpha|} r_i) M_{|\beta-k|}(\prod_{j=1}^{|\beta-k|} s_j) M_{|k|}(\prod_{j=1}^{|\alpha|} s_j)};\ \alpha, \beta \in \mathbf{N}_0^n \right\} \\
&\quad \le \sup\left\{ \left\| \sup_{\alpha\in\mathbf{N}_0}\left\{\frac{(1+|x|^2)^{\alpha/2}}{M_{|\alpha|}(\prod_{i=1}^{|\alpha|} r_i)}\right\} 2^{-\beta} \sum_{k\le\beta}\binom{\beta}{k} \sup_k\left\{\frac{|\partial^k(1/P_{\tilde{r}_p})|}{M_{|k|}(\prod_{j=1}^{|k|}(s_j/2))}\right\} \right.\right. \\
&\qquad\qquad \left.\left. \cdot \frac{|\partial^{\beta-k}\phi|}{M_{|k-\beta|}(\prod_{j=1}^{|\beta-k|}(s_j/2))} \right\|_{L^2};\ k, \beta \in \mathbf{N}_0^n,\ k \le \beta \right\} \\
&\quad \le D \sup\left\{ \left\| \exp N(c|x|) \sup_k \left\{\frac{|k|!d^{-k}}{M_{|k|}(\prod_{j=1}^{|k|}(s_j/2))}\right\} \right.\right. \\
&\qquad\qquad \left.\left. \cdot \exp(-N(|x|)/C) 2^{-\beta} \sum_{k\le\beta}\binom{\beta}{k} \frac{|\partial^{\beta-k}\phi|}{M_{|\beta-k|}(\prod_{j=1}^{|\beta-k|}(s_j/2))} \right\|_{L^2}; \right. \\
&\qquad\qquad \left. k, \beta \in \mathbf{N}_0^n, k \le \beta \right\} \\
&\quad \le C_1 \sup\left\{ 2^{-\beta} \sum_{k\le\beta}\binom{\beta}{k} \frac{\|\partial^{\beta-k}\phi\|_{L^2}}{M_{|\beta-k|}(\prod_{j=1}^{|\beta-k|}(s_j/2))},\ k, \beta \in \mathbf{N}_0,\ k \le \beta \right\} \\
&\quad \le C_1 \|\phi\|_{L^2, s_{j/2}}.
\end{aligned}$$

Thus, (15) implies that for suitable $C_1 > 0$

$$\begin{aligned}
|\langle \hat{f}(\xi)/P_{\tilde{r}_p}(\xi), \phi(\xi)\rangle| &= |\langle \hat{f}(\xi), \phi(\xi)/P_{\tilde{r}_p}(\xi)\rangle| \\
&\le C_1 \|\phi\|_{L^s, s_{p/2}}, \qquad f \in A,\ \phi \in \mathscr{D}_{L^2}^{\{M_p\}}.
\end{aligned}$$

This implies that $\{\hat{f}/P_{\tilde{r}_p};\ f \in A\}$ is equicontinuous in $\widetilde{\mathscr{D}}_{L^2}^{\prime\{M_p\}}$, and by Theorem 1 every $\hat{f}$ $(f \in A)$ is of the form

$$\hat{f}(\xi) = P_{\tilde{r}_p}(\xi)(P_{\tilde{\tilde{r}}_p}(D)\widetilde{F}_1(\xi) + \widetilde{F}_2(\xi)), \qquad \widetilde{F}_1 \in \widetilde{A}_1,\ \widetilde{F}_2 \in \widetilde{A}_2,$$

where $\widetilde{A}_1$ and $\widetilde{A}_2$ are bounded subsets of L^2. By the inverse Fourier transform we obtain

$$f(x) = P_{\tilde{r}_p}(D)(P_{\tilde{\tilde{r}}_p}(x)F_1(x) + F_2(x)), \qquad F_1 \in A_1,\ F_2 \in A_2,$$

where A_1 and A_2 are bounded subsets of L^2. Put

$$\begin{aligned}
F(x) = \int_0^{x_1} \cdots \int_0^{x_n} (P_{\tilde{\tilde{r}}_p}(t)F_1(t) + F_2(t))\, dt_1 \cdots dt_n, \\
x \in \mathbf{R}^n,\ F_1 \in A_1,\ F_2 \in A_2,
\end{aligned}$$

$$P(D) = P_{\tilde{r}_p}(D)\frac{\partial^n}{\partial x_1 \cdots \partial x_n}.$$

From (4) it follows that

$$|F(x)| \le C\exp(\tilde{\tilde{N}}(|x|))(1+|x|^2)^n \int_0^x \frac{F_1(t)+F_2(t)}{(1+|t|^2)^n}\,dt$$
$$\le C_1(\|F_1\|_{L^2}+\|F_2\|_{L^2})(1+|x|^2)^n \exp\tilde{\tilde{N}}(x), \qquad x\in\mathbf{R}^n.$$

Since for every $k>0$ there is $\rho_k>0$ such that [4]

$$\tilde{\tilde{N}}(x) \le M(k|x|), \qquad |x|>\rho_k,$$

(14) follows and the theorem is proved.

4. Bounded sets in $\mathscr{D}'^*$

In this section we shall give two theorems analogous to the corresponding theorems for distributions [12].

Lemma 6. *Let* $f\in\mathscr{D}'^*$, Ω *be a bounded open set, and* K *be a compact set. The mapping*

$$G_f\colon \mathscr{D}_K^*\times\mathscr{D}_K^*\to L^\infty(\Omega), \qquad (\alpha,\beta)\to G_f(\alpha,\beta)=f*\alpha*\beta_{|\Omega},$$

is continuous. (*Symbol* $_{|\Omega}$ *means the restriction on* Ω.)

Proof. Let $K_1=K+K=\{x+y;\ x,y\in K\}$ and Ω_1 be an open bounded set such that $\Omega_1\supset\overline{\Omega-K_1}$. From [4, Theorem 10.3] it follows that

$$f_{|\Omega_1}=P(D)F,$$

where F is continuous on $\overline{\Omega_1}$ and $P=P_r$ is of form (1) (resp. $P=P_{r_p}$ is of form (3)). Let $\varepsilon>0$ be given. There are $\tilde{r}>0$ (resp. $\tilde{r}_p\in\mathfrak{R}$) and $\delta>0$ such that

$$\alpha\in V_{\tilde{r}}(\delta)=\{\phi\in\mathscr{D}_K^{(M_p)};\ \|\phi\|_{K,\tilde{r}}<\delta\}\Rightarrow\sup_{x\in K}\{|P_r(D)\alpha(x)|\}<\varepsilon$$
$$\left(\alpha\in V_{\tilde{r}_p}(\delta)=\{\phi\in\mathscr{D}_K^{\{M_p\}};\ \|\phi\|_{K,\tilde{r}_p}>\delta\}\Rightarrow\sup_{x\in K}\{|P_{r_p}(D)\alpha(x)|\}<\varepsilon\right).$$

Let α, $\beta\in V_{\tilde{r}}(\delta)$ (resp. α, $\beta\in V_{\tilde{r}_p}(\delta)$) and $x\in\Omega$. In the (M_p)-case

$$\begin{aligned}|(f*\alpha*\beta)(x)| &\le |\langle P_r(D)F(t),\,(\alpha*\beta)(x-t)\rangle|\\ &= |\langle F(t),\,(P_r\alpha*\beta)(x-t)\rangle|\\ &\le \int_{\Omega_1}|F(t)(P_r\alpha*\beta)(x-t)|\,dt\\ &\le C\sup_{t\in\Omega_1}\{|(P_r(D)\alpha*\beta)(x-t)|\}\\ &\le C_1\sup_{t\in K}\{|P_r(D)\alpha(t)|\}\sup_{t\in K}\{|\beta(t)|\}\le C_1\varepsilon^2,\end{aligned}$$

where $C=\int_\Omega F(t)\,dt$, $C_1=C\cdot\operatorname{mes}\Omega$.

The same holds in the $\{M_p\}$ case. Thus, in both cases we obtain the continuity of G_f.

Theorem 3. *Let B' be a subset of $\mathscr{D}'^*$.*

(i) *B' is bounded in $\mathscr{D}'^*$ if and only if for every bounded open set $\Omega \subset \mathbf{R}^n$ and every $\phi \in \mathscr{D}^*$*

$$\sup\{|(T * \phi)(x)|\,;\, x \in \Omega,\ T \in B'\} < \infty. \tag{17}$$

(ii) *B' is bounded in $\mathscr{D}'^*$ if and only if for every open bounded set $\Omega \subset \mathbf{R}^n$ and every open bounded neighbourhood of zero ω there is $r > 0$ (resp. there is $r_p \in \mathfrak{R}$) such that for every $\phi \in \mathscr{D}^{M_p}_{\omega, r/2}$ (resp. for every $\phi \in \mathscr{D}^{\{M_p\}}_{\omega, r_p}$) (17) holds.*

Proof. (i) Let B' be a bounded set in $\mathscr{D}'^*$. Let $\phi \in \mathscr{D}^*$, $\operatorname{supp}\phi = K$, and Ω_1 and Ω_2 be open bounded sets in $\mathbf{R}^n$ such that $\Omega_1 \supset \overline{\Omega - K}$, $\Omega_2 \supset \overline{\Omega_1}$. The set B'_{Ω_2} of restrictions $T_{|\Omega_2}$, $T \in B'$, is bounded in $\mathscr{D}'^*(\Omega_2)$.

Consider the set $B = \{\phi_x\,;\, x \in \Omega\}$, where

$$\phi_x \colon t \to \phi(x - t), \qquad t \in \Omega_2,\ x \in \Omega.$$

All the elements of B have the supports contained in $\overline{\Omega_1}$. Moreover, B is bounded in $\mathscr{D}^*(\Omega_2)$. Because of that, for some $C > 0$,

$$|\langle T, \phi_x\rangle| = |(T * \phi)(x)| < C, \qquad T \in B',\ x \in \Omega.$$

Now, assume that, for every $\phi \in \mathscr{D}'^*$, (17) holds. As in Lemma 6, for every $T \in B'$ we denote by G_T the continuous linear mapping from $\mathscr{D}^*_K \times \mathscr{D}^*_K$ into $L^\infty(\Omega)$. The set $\mathscr{G} = \{G_T\,;\, T \in B'\}$ is pointwisely bounded. Namely, (17) implies that for fixed $\alpha \in \mathscr{D}^*_K$ the set $\{T * \alpha\,;\, T \in B'\}$ is bounded in $\mathscr{D}'^*_K$, and thus, for fixed $\beta \in \mathscr{D}^*_K$, there exist $C_\beta > 0$ such that

$$|G_T(\alpha, \beta)(x)| = |((T * \alpha) * \beta)(x)| \le C_\beta, \qquad G_T \in \mathscr{G},\ x \in \Omega.$$

Since $\mathscr{D}^*_K \times \mathscr{D}^*_K$ is barrelled, the family $\mathscr{G}$ is equicontinuous. This implies that there exists a neighbourhood of zero $V_r(\varepsilon)$ (resp. $V_{r_p}(\varepsilon)$) in $\mathscr{D}^{(M_p)}_K$ (resp. in $\mathscr{D}^{\{M_p\}}_K$) such that

$$\alpha, \beta \in V_r(\varepsilon) \Rightarrow \sup\{|(T * \alpha * \beta)(x)|\,;\ T \in B',\ x \in \Omega\} \le 1$$
$$(\text{resp. } \alpha, \beta \in V_{r_p}(\varepsilon) \Rightarrow \sup\{|(T * \alpha * \beta)(x)|\,;\ T \in B',\ x \in \Omega\} \le 1).$$

The same holds for α, β being in the closure $\overline{V_r(\varepsilon)}$ (resp. $\overline{V_{r_p}(\varepsilon)}$) in $\mathscr{D}^{M_p}_{K, r/2}$ (resp. $\mathscr{D}^{\{M_p\}}_{K, r_p}$).

This implies that for arbitrary θ, $\mu \in \mathscr{D}^{(M_p)}_{\omega, r/2}$ (resp. θ, $\mu \in \mathscr{D}^{\{M_p\}}_{\omega, r_p}$)

$$\sup\{|(T * \theta * \mu)(x)|\,;\ T \in B',\ x \in \Omega\} < \infty.$$

By using two times (9) (resp. (10)) we have that on Ω

$$T = P_r(D)P_r(D)(T * u * u) - P_r(D)(T * u * \psi) - T * \psi, \qquad T \in B'$$

(resp. $T = P_{r_p}(D)P_{r_p}(D)(T * u * u) - P_{r_p}(D)(T * u * \psi) - T * \psi$, $T \in B'$). Because $u \in \mathscr{D}^{\{M_p\}}_{\omega, r/2}$ and $\psi \in \mathscr{D}^{(M_p)}_{\omega}$ (resp. $u \in \mathscr{D}^{\{M_p\}}_{\omega, r_p}$ and $\psi \in \mathscr{D}^{\{M_p\}}_K$), sets $\{T * u * u\,;\, T \in B'\}$, $\{T * u * \psi\,;\, T \in B'\}$, and $\{T * \psi\,;\ t \in B'\}$ are bounded in $L^\infty(\Omega)$. This implies that $\sup\{|\langle T, \phi\rangle\}|\,;\, T \in B'\} < \infty$ for every $\phi \in \mathscr{D}^*$ with $\operatorname{supp}\phi \subset \Omega$. Since Ω is an arbitrary open bounded set in $\mathbf{R}^n$, it follows that B' is bounded in $\mathscr{D}'^*$.

(ii) We only have to prove that for B' being bounded in $\mathscr{D}'^*$ the assertion in (ii) holds. By [4, Theorem 10.3] there is a set B of continuous functions uniformly bounded on $\overline{\Omega+\omega}$ and P_r (resp. P_{r_p}) such that every $T \in B'$ is of the form $T = P_r F_{|(\Omega+\omega)}$ (resp. $T = P_{r_p} F_{|(\Omega+\omega)}$) for some $F \in B$. There exists $\tilde{r}$ (resp. $\tilde{r}_p \in \mathfrak{R}$) such that $P_r(D)\phi$ is continuous if $\phi \in \mathscr{D}^{\{M_p\}}_{\omega,\tilde{r}/2}$ (resp. $P_{r_p}(D)\phi$ is continuous if $\phi \in \mathscr{D}^{\{M_p\}}_{\omega,\tilde{r}/p}$).

Since, for $T \in B'$, $\phi \in \mathscr{D}^{M_p}_{\omega,\tilde{r}/2}$ (resp. $\phi \in \mathscr{D}^{\{M_p\}}_{\omega,\tilde{r}_p}$), $T * \phi = F * P_r(D)\phi$ (resp. $T * \phi = F * P_{\tau_p}(D)\phi$), $x \in \Omega$, the assertion follows.

Theorem 4. *Let f_ν be a sequence in $\mathscr{D}'^*$. Then:*

(i) *f_ν converges to 0 in $\mathscr{D}'^*$ if and only if, for every $\phi \in \mathscr{D}^*$ and every open bounded set $\Omega \in \mathbf{R}$, $f_\nu * \phi$ converges to 0 uniformly on Ω.*

(ii) *f_ν converges to 0 in $\mathscr{D}'^*$ if and only if for every open and bounded set $\Omega \subset \mathbf{R}^n$ and every open bounded neighbourhood of zero ω there is $r > 0$ (resp. there is $r_p \in \mathfrak{R}$) such that for every $\phi \in \mathscr{D}^{M_p}_{\omega,r/2}$ (resp. $\phi \in \mathscr{D}^{\{M_p\}}_{\omega,r_p}$) $f_\nu * \phi$ converges to 0 uniformly on Ω.*

We omit the proof but only remark that, in order to prove this, one has to use the equicontinuity of the mapping

$$\mathscr{D}^*_K \times \mathscr{D}^*_K \ni \alpha, \beta \to f_\nu * \alpha * \beta \in L^\infty(\Omega) \qquad (\nu \in \mathbf{N})$$

and that for every fixed α, β, $f_\nu * \alpha * \beta$ converges to 0 uniformly on Ω.

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