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RAMSEY-THEORY AND FORCING EXTENSIONS

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ABSTRACT. In every forcing extension of every model of set theory by a nontrivial (set) forcing there exist a graph Y and a cardinal μ, such that every graph has an edge coloring with μ colors with no monochromatic induced copy of Y.

One of the basic questions in the Ramsey theory of graphs is the following. Given the graphs X and Y and a cardinal μ, is it true that whenever the edges of X are μ-colored, there always exists a monochromatic induced copy of Y? We denote the affirmative answer by $X \to (Y)^2_\mu$, the negative one by $X \not\to (Y)^2_\mu$. The question whether for each Y, μ there exists such an X was answered positively for Y, μ finite in the early seventies. It was natural to assume that the answer is positive in the unrestricted case, too. However, in [2], Hajnal and Komjáth showed that adding a Cohen real produces a counterexample, a graph Y of cardinal ω_1 such that for no graph X in the enlarged model does $X \to (Y)^2_2$ hold. Soon after, S. Shelah showed that the "yes" answer is consistent, too; a class forcing gives a model, where for every pair Y, μ an appropriate X exists.

Here we show that there is nothing special about the Cohen forcing: every (set) forcing adds a counterexample, assuming that it adds something.

For a good exposition and more historical details we refer to [1].

We call a poset *nontrivial*, if every condition has incompatible extensions.

Theorem. *If $(P, \leq)$ is any nontrivial (set) notion of forcing, then, in V^P, there exist a graph Y and a cardinal μ such that $X \not\to (Y)^2_\mu$ holds for every graph X.*

Fix the poset $(P, \leq)$ for the rest of the paper.

Lemma 1. *There is a dense set $D = \{p(\alpha): \alpha < \kappa\} \subseteq P$ for some cardinal κ such that $p(\alpha) \not< p(\beta)$ whenever $\alpha < \beta < \kappa$.*

This is Hausdorff's theorem for partially ordered sets. For the sake of completeness we supply a proof.

Proof. Let κ be the minimal cardinal such that there is a dense set of cardinal κ, say $E = \{q(\alpha): \alpha < \kappa\}$. Put $\alpha \in X$ if for no $\beta < \alpha$ does $q(\beta) < q(\alpha)$ hold.

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We claim that $D = \{q(\alpha): \alpha \in X\}$ is dense. As this implies that $|D| = \kappa$, this clearly suffices. Assume that $p \in P$ is arbitrary. Let $\alpha < \kappa$ be the least ordinal such that $q(\alpha) \leq p$ (exists as E is dense). $\alpha \in X$, as otherwise, for some $\beta < \alpha$, $q(\beta) < q(\alpha) \leq p$, so α was not minimal. If now $X = \{\xi(\alpha): \alpha < \kappa\}$ is the increasing enumeration of X, then $p(\alpha) = q(\xi(\alpha))$ is as required.

Lemma 2. *There is a function $T: [2^\kappa]^2 \to \kappa$ such that whenever $h: 2^\kappa \to \kappa$, there is an $i < \kappa$ such that T assumes every value $< \kappa$ on $h^{-1}(\{i\})$.*

Proof. In the proof we repeatedly use König's theorem, i.e., that $\mathrm{cf}(2^\kappa) > \kappa$. As $(2^\kappa)^\kappa = 2^\kappa$ it is possible to define $T: [2^\kappa]^2 \to \kappa$ with the following property. If $x(\alpha) < 2^\kappa$ are different and $i(\alpha) < \kappa$ for $\alpha < \kappa$, then there exist arbitrary large $y < 2^\kappa$ such that $T(x(\alpha), y) = i(\alpha)$ for all $\alpha < \kappa$. We show that this T works.

Assume that $h: 2^\kappa \to \kappa$ is a counterexample and that the color class $h^{-1}(\{\alpha\})$ misses some $i(\alpha) < \kappa$. Put $S = \{\alpha: \sup(h^{-1}(\{\alpha\})) = 2^\kappa\}$. Select $z < 2^\kappa$ so large that $\sup(h^{-1}(\{\alpha\})) < z$ for $\alpha \notin S$ and $h^{-1}(\{\alpha\})$ has at least κ elements below z for $\alpha \in S$. Then we can choose different $x(\alpha) < z$, $h(x(\alpha)) = \alpha$ ($\alpha \in S$) and find $y > z$ with $T(x(\alpha), y) = i(\alpha)$ (by the way T was constructed). If now $h(y) = \alpha$, clearly $\alpha \in S$, so then $h^{-1}(\{\alpha\})$ does not miss color $i(\alpha)$, a contradiction.

Given $\{p(\alpha): \alpha < \kappa\}$ as in Lemma 1 and T as in Lemma 2, assume that $G \subseteq P$ is a generic set. We construct the graph Y in the theorem as follows. The vertex set will be 2^κ. For $\alpha < \beta < 2^\kappa$ we let $\{\alpha, \beta\}$ be an edge of Y iff $p(T(\alpha, \beta)) \in G$. Assume that $1 \Vdash X$ is a graph on λ. In $V[G]$, if $\alpha < \beta < \lambda$ and $\{\alpha, \beta\} \in X$, then some element of $D \cap G$ forces this, as D is dense, and those conditions determining the truth value of $\{\alpha, \beta\} \in X$ form a dense, open set, so G meets the intersection of these two sets and the condition in the intersection cannot force $\{\alpha, \beta\} \notin X$. Color the edge $\{\alpha, \beta\}$ by the least $\xi < \kappa$ such that $p(\xi) \in G$ forces $\{\alpha, \beta\} \in X$.

Assume that there are $p \in P$, $i < \kappa$ such that $p \Vdash f: 2^\kappa \to \lambda$ embeds Y into the ith color of X. Clearly, p and $p(i)$ are compatible (as otherwise p forces that no edge gets color i), so we may as well assume that $p \leq p(i)$. For every $\alpha < 2^\kappa$ select $\gamma(\alpha) < \kappa$, $g(\alpha) < \lambda$ such that $p(\gamma(\alpha)) \leq p$, $p(\gamma(\alpha)) \Vdash f(\alpha) = g(\alpha)$. By Lemma 2, there is a $j < \kappa$ such that T assumes every value $< \kappa$ in $\{\alpha < 2^\kappa: \gamma(\alpha) = j\}$. Select incompatible $p(\delta)$, $p(\delta') < p(j)$. As $p(\delta) < p(j) \leq p \leq p(i)$, by Lemma 1, $i \leq j < \delta$ holds. There are $\alpha < \beta < 2^\kappa$ with $\gamma(\alpha) = \gamma(\beta) = j$, $T(\alpha, \beta) = \delta$. Then $p(\delta)$ forces that $\{\alpha, \beta\} \in Y$, $f(\alpha) = g(\alpha)$, $f(\beta) = g(\beta)$ which imply that $\{g(\alpha), g(\beta)\} \in X$. By the way we colored the edges of X, $p(i) \Vdash \{g(\alpha), g(\beta)\} \in X$, which cannot be true as $p(\delta')$ forces that $f(\alpha) = g(\alpha)$, $f(\beta) = g(\beta)$, but it also forces that $p(\delta) \notin G$ and so $\{\alpha, \beta\} \notin Y$, so $\{g(\alpha), g(\beta)\} \notin X$.

References

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