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ANTICOMMUTANT LIFTING AND ANTICOMMUTING DILATION

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ABSTRACT. In this paper we prove the anticommutant counterpart of the Sz.-Nagy-Foiaş commutant lifting theorem and Ando's theorem on unitary dilation of a pair of commutative contractions. A dual extension theorem applies in the one-step procedure of our approach.

1. INTRODUCTION

One of the best known results in dilation, extension, and interpolation theory is the Sz.-Nagy-Foiaş commutant lifting theorem [8]. Ando's theorem [1] on commuting unitary dilation of a pair of commutative contractions is also a consequence of the theorem mentioned above [3, p. 321].

In Theorems 2 and 3 we prove the anticommutant counterparts of the Sz.-Nagy-Foiaş commutant lifting theorem. Our proof includes both variants.

Our approach applies each step of the dual extension version of Parrot's theorem [3, Theorem 1], one of the neatest results in extension, dilation, and interpolation theory (see, e.g., [2, 4, 5]).

2. DUAL EXTENSION

We formulate here a dual extension version of Parrott's theorem on quotient norms [3, Theorem 1] in a different way than Foiaş and Tannenbaum did [2, Theorem 1] (see [4, Theorem] or [5, Theorem 1]).

Theorem 1. *Let K and K' be Hilbert spaces, $H \subset K$ and $H' \subset K'$ be subspaces, and $X: H \to K'$ and $X': H' \to K$ be given bounded linear transformations. Then there exists operator $Y: K \to K'$ extending X so that Y^* extends X' if and only if the following identity holds true*:

$$(Xh, h') = (h, X'h') \qquad (h \in H, \ h' \in H').$$

Moreover Y can be of norm $\max\{\|X\|, \|X'\|\}$ *possible at most.*

Proof. Reduction to a selfadjoint norm-preserving extension theorem of Krein will be as follows. Given the symmetric operator $S: H \oplus H' \to K \oplus K'$ by

$$S(h; h') := (X'h'; Xh) \qquad (h \in H, \ h' \in H')$$

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we see that $\|S\| = \max\{\|X\|, \|X'\|\}$. So S has selfadjoint extension $\widetilde{S}$ to $K \oplus K'$ such that $\|\widetilde{S}\| = \|S\|$. Then $Y = P'\widetilde{S}P^*$ is a solution of this dual extension problem with extremal (least) norm where P and P' are the natural projections of $K \oplus K'$ onto K and K', respectively. The extension properties of Y are plain and the norm of Y not greater than $\|\widetilde{S}\| = \max\{\|X\|, \|X'\|\}$ on one side but at least such otherwise.

3. (ANTI-)COMMUTANT LIFTING

Given a Hilbert space H, an operator S on H will be called a contraction if its norm ≤ 1. And the operator X_0 on H belongs to the commutant or anticommutant of S if X_0 satisfies for $\mathscr{E} = 1$ or -1, respectively,

$$SX_0 = \mathscr{E} X_0 S. \tag{1}$$

Now let U be the minimal isometric dilation of S acting on a Hilbert space K, containing H as a subspace. We know [3, 8] that U^* extends S^* and $K = \bigcup_{n=0}^{\infty} K_n$ with $K_n = \bigvee_{k=0}^{n} U^k(H)$ (thus $K_0 = H$), moreover the orthogonal projections $P_n \colon K \to K_n$ satisfy $P_{n+1}U = UP_n$ for any natural n.

Let X_0 be an anticommutant or commutant operator of S. We lift X_0 to K in the minimal unitary dilation space.

Theorem 2. *Let S be a contraction and X_0 be an operator on the Hilbert space H. Assume identity* (1). *If U is the minimal isometric dilation of S acting on the Hilbert space K then there exists an operator $X_{\mathscr{E}}$ on K such that $X_{\mathscr{E}}^*$ extends X_0^*, $\|X_{\mathscr{E}}\| = \|X_0\|$ and*

$$UX_{\mathscr{E}} = \mathscr{E} X_{\mathscr{E}} U. \tag{2}$$

Proof. At the first step we find $X_1 \colon K_1 \to K_1$ with $X_1^*|_{K_0} = X_0^*$, $X_1|_{H_0} = \mathscr{E} U X_0 U^*|_{H_0}$, where $H_0 = U(H)$, and such that $\|X_1\| = \|X_0\|$ and $\mathscr{E} U X_0 P_0 = X_1 P_1 U$. To do this we should check the adjoining identity of Theorem 1: for $h, k \in H_0$

$$\begin{aligned}(\mathscr{E} U X_0 U^*(Uh), k) &= \mathscr{E}(X_0 h, U^* k) = \mathscr{E}(X_0 h, S^* k) = \mathscr{E}(S X_0 h, k)\\ &\overset{(1)}{=} \mathscr{E}^2(X_0 S h, k) = (P_0 S h, X_0^* k) = (Uh, X_0^* k).\end{aligned}$$

This solution then satisfies

$$\mathscr{E} U X_0 P_0 = \mathscr{E} U X_0 U^*(U P_0) = X_1(U P_0) = X_1 P_1 U.$$

The second step yields $X_2 \colon K_2 \to K_2$ such that $X_2^*|_{K_1} = X_1^*$, $X_2|_{H_1} = \mathscr{E} U X_1 U^*|_{H_1}$, where $H_1 = U(K_1)$, and moreover $\|X_2\| = \|X_1\|$ and $\mathscr{E} U X_1 P_1 = X_2 P_2 U$. That the adjoining identity satisfies in this setting is less obvious:

$$\begin{aligned}(\mathscr{E} U X_1 U^*(Uh), k) &= \mathscr{E}(P_1 U X_1 h, k) = \mathscr{E}(U P_0 X_1 h, k)\\ &= \mathscr{E}(h, X_1^* P_0 U^* k) = \mathscr{E}(h, X_0^* P_0 U^* k)\\ &= (\mathscr{E} U X_0 P_0 h, k) = (X_1 P_1 U h, k)\\ &= (Uh, P_1 X_1^* k) = (Uh, X_1^* k)\end{aligned}$$

holds indeed for any $h, k \in K_1$.

In the nth step we argue similarly to have $X_n \colon K_n \to K_n$ with $X_n^*|_{K_{n-1}} = X_{n-1}^*$, $X_n|_{H_{n-1}} = \mathscr{E} U X_{n-1} U^*|_{H_{n-1}}$, where $H_{n-1} = U(K_{n-1})$, and $\|X_n\| = \|X_{n-1}\|$, $\mathscr{E} U X_{n-1} P_{n-1} = X_n P_n U$.

The necessary change in checking is that one writes $n-1$ and $n-2$ instead of 1 and 0, respectively, used at the 2nd step. As a result of this process, thanks to the minimality of the dilation, we arrive at an operator $X_{\mathscr{E}}^*: K \to K$ as the extension of each of $\{X_n^*\}_{n=0}^{\infty}$, with norm $\|X_0\|$. Since then $X_n P_n$ converges to $X_{\mathscr{E}}$ in the strong operator topology, the identity $\mathscr{E} U X_{n-1} P_{n-1} = X_n P_n U$ gives the desired identity (2). Thus X_0 is lifted to a desired dilation $X_{\mathscr{E}}$ on K.

Theorem 3. *Let S be a contraction and X_0 be an operator on the Hilbert space H with identity* (1). *If U is the minimal unitary dilation of S acting on a Hilbert space K then there exists an operator $X_{\mathscr{E}}$ on K which is the dilation of X_0, $\|X_{\mathscr{E}}\| = \|X_0\|$, and identity* (2) *also holds true.*

Proof. First let U_+ be the minimal isometric dilation of S acting on a Hilbert space K_+ that lies between H and K. An application of Theorem 2 gives a dilation $X_{\mathscr{E}+}$ of X to K_+ with $\|X_{\mathscr{E}+}\| = \|X\|$ and the corresponding identity holds:

$$(2_+) \qquad U_+ X_{\mathscr{E}+} = \mathscr{E} X_{\mathscr{E}+} U_+.$$

Now, as U^* is the minimal isometric dilation of U_+^*, identity (2_+) reads as follows:

$$U_+^* X_{\mathscr{E}+}^* = \mathscr{E} X_{\mathscr{E}+}^* U_+^*.$$

Another application of Theorem 2 yields an operator $X_{\mathscr{E}}^*$ on K that extends $X_{\mathscr{E}+}$, $\|X_{\mathscr{E}}\| = \|X_{\mathscr{E}+}\| = \|X_0\|$, and identity (2) satisfies. The proof is complete.

4. (Anti-)commuting dilation

The situation changes to a pair of contractions S and T on a Hilbert space H, which satisfies

$$(3) \qquad ST = \mathscr{E} TS, \quad \text{where } \mathscr{E} = -1 \text{ or } 1.$$

In other words S, T are (anti-)commuting operators if $\mathscr{E}$ equals (-1) or 1. Ando's theorem [1] ensures for the $\mathscr{E} = 1$ case a commuting pair of unitaries U, V on a larger Hilbert space so that U and V are dilations of S and T, respectively. Here we prove the similar statement for anticommuting contractions and unitaries as well.

Theorem 4. *Let S and T be* (*anti-*)*commuting contractions on a Hilbert space H. Then there exists an* (*anti-*)*commuting pair of unitaries U and V that are common unitary dilations of S and T, respectively.*

Proof. An application of Theorem 3 yields a dilation of the pair S, T, say, U_0, T_0, respectively, acting on a Hilbert space K_0 so that U_0 is the minimal unitary dilation of S, and T_0 is a contractive dilation of T to K_0 with

$$U_0 T_0 = \mathscr{E} T_0 U.$$

Then let V be the minimal unitary dilation of T_0 that acts on a Hilbert space K. Our goal is to extend U_0 to a unitary operator U on K so that

$$UV = \mathscr{E} VU.$$

Due to the minimality assumption $K = \bigvee_{n=-\infty}^{\infty} V^n(K_0)$ the operator U has to be defined on $V^n k_0$ for $n \in \mathbb{Z}$ and $k_0 \in K_0$ as $\mathscr{E}^n V^n U_0 k_0$. Doing so, U is seen to be an onto map. That U is isometric, hence unitary, follows by the identity (if $m \geq n$)

$$\begin{aligned}(\mathscr{E}^m V^m U_0 h_0, \mathscr{E}^n V^n U_0 k_0) &= \mathscr{E}^{m+n}(V^{m-n} U_0 h_0, U_0 k_0) \\ &= \mathscr{E}^{m+n}(T_0^{m-n} U_0 h_0, U_0 k_0) \\ &= \mathscr{E}^{m+n}\mathscr{E}^{m-n}(U_0 T_0^{m-n} h_0, U_0 k_0) \\ &= (T_0^{m-n} h_0, k_0) = (V^m h_0, V^n k_0),\end{aligned}$$

where $m, n \in \mathbb{Z}$, $h_0, k_0 \in K_0$, are arbitrary. Remark that K_0 reduces U as well. Our final goal is to prove

$$S^m T^n = P_H U^m V^n|_H \qquad (m, n = 0, 1, 2, \ldots),$$

where P_H stands for the orthogonal projection of K to H. First we know that

$$S^m T^n = (P_H U_0^m|_H)(P_H T_0^n|_H) = P_H U_0^m T_0^n|_H,$$

as in the proof of [9, Theorem 6]. Hence we have that for $h_0, k_0 \in K_0$ and $m, n \in N$

$$\begin{aligned}(U_0^m T_0^n h_0, k_0) &= (T_0^n h_0, U_0^{*m} k_0) = (V^n h_0, U_0^{*m} k_0) \\ &= (V^n h_0, U^{*m} k_0) = (U^m V^n h_0, k_0).\end{aligned}$$

This implies that the (anti-)commuting pair of unitaries U, V are common dilations of the pair of contractions S, T indeed.

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