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ON DIRICHLET SERIES ASSOCIATED WITH POLYNOMIALS

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ABSTRACT. Let $P(X)$ be a polynomial of degree N with complex coefficients and d_1, d_2 two complex numbers with real part greater then -1. Consider the Dirichlet series associated with the triple $(P(X), d_1, d_2)$

$$L(s) = \sum_{n=1}^{\infty} \frac{P(n)}{(n+d_1)^s(n+d_2)^s}.$$

In this paper we get an explicit formula for $L(s)$ in terms of special functions which gives meromorphic continuation of $L(s)$ with at most simple poles at $s = (N+1-k)/2$, $k = 0, 1, \ldots$. Finally we apply our explicit formula to Minakshisundaram's zeta function of the three-dimensional sphere.

1. INTRODUCTION

Mellin first studied the analytic continuation of Dirichlet series associated with one polynomial of several variables in [6] (see also [3]). We are concerned with Dirichlet series of the form $\sum_{n=0}^{\infty} P(n)Q(n)^{-s}$, where $P(n)$ and $Q(n)$ are two polynomials of one variable. The aim of this paper is to obtain meromorphic continuation of Dirichlet series of the form

$$L(s) = \sum_{n=1}^{\infty} \frac{P(n)}{(n+d_1)^s(n+d_2)^s} \tag{1}$$

where d_1, d_2 are two complex numbers with $\Re(d_i) > -1$, $i = 1, 2$, and $P(n)$ is a polynomial of degree N. The above series converges absolutely in the half plane $\Re(s) > (N+1)/2$, where it represents an analytic function. In [8] Eie obtains an analytic continuation for Dirichlet series of the type:

$$\sum_{n=1}^{\infty} \frac{1}{[(n+d_1)(n+d_2)\cdots(n+d_k)]^s}$$

by giving an integral expression (see [8, p. 586]). In our paper after obtaining a similar integral expression we succeed in getting an explicit formula by means of special functions. This gives the analytic continuation.

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Remarkable examples of our Dirichlet series are Minakshisundaram's zeta functions $\sum_{n=1}^{\infty} \lambda_n^{-s}$, where the λ_n 's are the eigenvalues of the Laplace operator on the canonical spaceforms (real spheres, real and complex projective spaces), (see [2, pp. 146–178]) so that we obtain a proof of their analytic continuation involving only tools of complex analysis (for different methods see [7, 4]).

We get the following result.

Theorem. *Let* $L(s)$ *be defined by* (1), *where* d_1, d_2 *are two complex numbers with* $\Re(d_i) > -1$, $i = 1, 2$, *and* $P(n)$ *is a polynomial of degree* N. *Then* $L(s)$ *has a meromorphic continuation on the complex plane with at most simple poles at* $s = (N+1-k)/2$, $k = 0, 1, 2, \ldots$.

We deduce our result from formula (5) which is in turn obtained by a direct computation. The key point in our proof is the lemma below that generalizes a well-known formula (see [5, no. 3411-7]). Finally in §3 we apply (5) to Minakshisundaram's zeta function of the sphere S^3.

2. Proof of the theorem

Lemma. *Let* m *be a nonnegative integer,* α *and* β *be complex numbers such that* $\Re(\alpha) > m$, $\Re(\beta) < m+1$, *and*

$$I(\alpha, \beta, m) = \int_0^{\infty} \frac{e^{\beta t} t^{\alpha}}{(e^t - 1)^{m+1}}\, dt.$$

Then

$$I(\alpha, \beta, m) = \Gamma(\alpha+1) \sum_{j=0}^{m} P_{m-j}^{m}(\beta) \zeta(\alpha + 1 - j,\ m - \beta + 1)$$

where $P_{m-j}^{m}(\beta)$ *are polynomials in* β *of degree* $m - j$ *defined by*

$$P_0^0(\beta) = 1, \qquad P_m^m(\beta) = \frac{1}{m!} \prod_{k=1}^{m} (\beta - k),$$

$$P_{m-j}^{m}(\beta) = \frac{1}{j!} \frac{d^j}{d\beta^j} P_m^m(\beta), \qquad 0 \le j \le m,$$

and $\zeta(s, a)$ *is the Hurwitz zeta function.*

Proof. $I(\alpha, \beta, m)$ is clearly absolute convergent, and integrating by parts we have for $m \ge 1$ that

$$I(\alpha, \beta, m) = \tfrac{1}{m}[\alpha I(\alpha - 1, \beta - 1, m - 1) + (\beta - 1) I(\alpha, \beta - 1, m - 1)].$$

Since

$$I(\alpha, \beta, 0) = \Gamma(\alpha + 1) \zeta(\alpha + 1, 1 - \beta)$$

(see [5, no. 3411-7]), the result follows by induction. □

We observe that

$$m! P_m^m(\beta) = \sum_{k=1}^{m+1} s_{m+1,k} \beta^{k-1}$$

where $s_{m+1,k}$ are the Stirling numbers of the first kind (see [1, Chapter III, pp. 88–93]).

Now we prove the theorem. Let us consider the series

$$L_m(s) = \sum_{n=1}^{\infty} \frac{n^m}{(n+d_1)^s(n+d_2)^s}.$$

It converges absolutely (and uniformly on compact subsets) in $\Re(s) > (1+m)/2$. If $P(n) = \sum_{m=0}^{N} a_m n^m$, we have

$$L(s) = \sum_{m=0}^{N} a_m L_m(s). \tag{2}$$

Starting from well-known formulas

$$\int_0^\infty y_i^{s-1} e^{-(n+d_i)y_i}\, dy_i = \Gamma(s)(n+d_i)^{-s}, \qquad i = 1, 2,$$

by multiplying these integrals, and summing on n we get

$$\begin{aligned} L_m(s)\Gamma(s)^2 &= \int_0^\infty \int_0^\infty \sum_{n=1}^{\infty} n^m e^{-n(y_1+y_2)} (y_1 y_2)^{s-1} e^{-d_1 y_1 - d_2 y_2}\, dy_1\, dy_2 \\ &= \int_0^\infty \int_0^\infty \frac{P_m(e^{-(y_1+y_2)})}{(1-e^{-(y_1+y_2)})^{m+1}} (y_1 y_2)^{s-1} e^{-d_1 y_1 - d_2 y_2}\, dy_1\, dy_2. \end{aligned}$$

In fact (see [1, Chapter V]), we have $\sum_{n=1}^{\infty} n^m z^n = P_m(z)/(1-z)^{m+1}$ where $P_0(z) = z$ and, if $m \neq 0$, $P_m(z) = \sum_{p=1}^{m} a_p^m z^p$ with

$$a_p^m = \sum_{l=1}^{p} (-1)^{p-l} \binom{m+1}{p-l} l^m.$$

Now we use the substitution

$$\begin{cases} y_1 = t_1 t_2, & t_1 \in (0, +\infty), \\ y_2 = t_1(1-t_2), & t_2 \in (0, 1) \end{cases}$$

so that, setting $\delta = d_2 - d_1$ and $\Delta = (d_2 + d_1)/2$, by [5, no. 3383-2], we get

$$\begin{aligned} L_m(s)\Gamma(s)^2 &= \int_0^\infty \int_0^1 \frac{P_m(e^{-t_1})}{(1-e^{-t_1})^{m+1}} t_1^{2s-1} t_2^{s-1} (1-t_2)^{s-1} e^{\delta t_1 t_2 - d_2 t_1}\, dt_1\, dt_2 \\ &= \int_0^\infty \frac{P_m(e^{-t_1})}{(1-e^{-t_1})^{m+1}} e^{-d_2 t_1} t_1^{2s-1} \int_0^1 e^{\delta t_1 t_2} t_2^{s-1} (1-t_2)^{s-1}\, dt_2\, dt_1 \\ &= \sqrt{\pi} \int_0^\infty \frac{P_m(e^{-t_1})}{(1-e^{-t_1})^{m+1}} t_1^{2s-1} t_1^{1/2-s} \delta^{1/2-s} e^{-d_2 t_1 + t_1\delta/2} \\ &\quad \times \Gamma(s) I_{s-1/2}\left(\frac{t_1 \delta}{2}\right) dt_1, \end{aligned}$$

where $I_{s-1/2}(z)$ is the Bessel function of the first kind. Hence by [5, no. 8445]

and the lemma, for $m \geq 1$ we have that
(3)

$$
\begin{aligned}
L_m(s)\Gamma(s) &= \sqrt{\pi}\delta^{1/2-s} \int_0^\infty \frac{P_m(e^{-t})}{(1-e^{-t})^{m+1}} t^{s-1/2} e^{-d_2 t+\delta t/2} I_{s-1/2}\left(\frac{t\delta}{2}\right) dt \\
&= \sqrt{\pi}4^{1/2-s} \int_0^\infty \frac{P_m(e^{-t})}{(1-e^{-t})^{m+1}} t^{2s-1} e^{-d_2 t+\delta t/2} \sum_{l=0}^\infty \left(\frac{\delta}{4}\right)^{2l} \frac{t^{2l}}{l!\Gamma(s+l+1/2)} dt \\
&= \sqrt{\pi}4^{1/2-s} \sum_{l=0}^\infty \left(\frac{\delta}{4}\right)^{2l} \frac{1}{l!\Gamma(s+l+1/2)} \int_0^\infty \frac{P_m(e^{-t})}{(e^t-1)^{m+1}} t^{2s+2l-1} e^{t(m+1-\Delta)} dt \\
&= \sqrt{\pi}4^{1/2-s} \sum_{l=0}^\infty \left(\frac{\delta}{4}\right)^{2l} \frac{1}{l!\Gamma(s+l+1/2)} \sum_{p=1}^m a_p^m \int_0^\infty \frac{e^{t(m+1-\Delta-p)}}{(e^t-1)^{m+1}} t^{2s+2l-1} dt \\
&= \sqrt{\pi}4^{1/2-s} \sum_{j=0}^m \sum_{p=1}^m \sum_{l=0}^\infty \left(\frac{\delta}{4}\right)^{2l} \frac{\Gamma(2s+2l)a_p^m}{l!\Gamma(s+l+1/2)} P_{m-j}^m(m+1-p-\Delta) \\
&\qquad \times \zeta(2s+2l-j,\ p+\Delta) \\
&= \sum_{j=0}^m \sum_{p=1}^m \sum_{l=0}^\infty \left(\frac{\delta}{2}\right)^{2l} \frac{\Gamma(s+l)a_p^m}{l!} P_{m-j}^m(m+1-p-\Delta) \\
&\qquad \times \zeta(2s+2l-j,\ p+\Delta).
\end{aligned}
$$

If $m = 0$, we have

$$
\begin{aligned}
L_0(s)\Gamma(s) &= \sqrt{\pi}4^{1/2-s} \sum_{l=0}^\infty \left(\frac{\delta}{4}\right)^{2l} \frac{1}{l!\Gamma(s+l+1/2)} \int_0^\infty \frac{e^{-\Delta t}}{(e^t-1)} t^{2s+2l-1} dt \\
&= \sqrt{\pi}4^{1/2-s} \sum_{l=0}^\infty \left(\frac{\delta}{4}\right)^{2l} \frac{\Gamma(2s+2l)}{l!\Gamma(s+l+1/2)} \zeta(2s+2l,\ 1+\Delta).
\end{aligned} \tag{4}
$$

Hence from (2)–(4) and well-known formulas for the Γ-function we get

$$
\begin{aligned}
L(s) &= \sum_{l=0}^\infty \left(\frac{\delta}{2}\right)^{2l} (-1)^l \binom{-s}{l} \\
&\quad \times \Bigg[a_0\zeta(2s+2l,\ 1+\Delta) \\
&\qquad + \sum_{m=1}^N \sum_{j=0}^m \sum_{p=1}^m a_m a_p^m P_{m-j}^m(m+1-p-\Delta)\zeta(2s+2l-j,\ p+\Delta)\Bigg].
\end{aligned} \tag{5}
$$

The above formula and the meromorphic continuation of Hurwitz zeta function give the meromorphic continuation over $\mathbb{C}$ of $L(s)$ with simple poles at most at $s = (N+1-k)/2$, where k runs through nonnegative integers, since

$$
\sum_{j=0}^m \sum_{l=0}^\infty \left(\frac{\delta}{2}\right)^{2l} (-1)^l \binom{-s}{l} \zeta(2s+2l-j,\ p+\Delta)
$$

is an absolutely (uniformly on compact subsets) convergent series on the set $\mathbb{C}\backslash\{(N+1-k)/2\,,\ k\in\mathbb{Z}^+\}$.

3. An application

Let us denote Minakshisundaram's zeta function of S^3 by $Z_3(s)$ so that, according to [2],

$$Z_3(s)=\sum_{n=1}^{\infty}\frac{n^2+2n+1}{[n(n+2)]^s}.$$

The values of parameters for $Z_3(s)$ in (5) are

$$N=2\,,\ 1\le m\le 2\,,\ a_0=1\,,\ a_1=2\,,\ a_2=1\,,\ a_1^1=a_2^2=a_1^2=1\,,$$
$$d_1=0\,,\ d_2=2\,,\ \tfrac{1}{2}\delta=1\,,\ \Delta=1\,,$$
$$P_1^1(\beta)=\beta-1\,,\ P_0^1(\beta)=1\,,\ P_2^2(\beta)=\tfrac{1}{2}(\beta-1)(\beta-2)\,,$$
$$P_1^2(\beta)=\beta-\tfrac{3}{2}\,,\ P_0^2(\beta)=\tfrac{1}{2}.$$

By reducing all Hurwitz zeta functions to $\zeta(-,2)$ a straightforward computation gives

$$Z_3(s)=\sum_{l=0}^{\infty}(-1)^l\binom{-s}{l}\zeta(2s+2l-2\,,\,2).$$

From above it follows that the poles of $Z_3(s)$ are exactly the points $s=\frac{3}{2}-n\,,\ n=0\,,1\,,\dots$, and they are all simple. Moreover,

$$\operatorname{Res}\left(Z_3(s)\,,\ \frac{3}{2}-n\right)=(-1)^n\frac{(2n-3)!!}{2^n n!}.$$

One should compare this way to obtain the residues of $Z_3(s)$ with that given in the general case (see [4, p. 79]). Finally we note that $Z_3(-n)=0\,,\ n=0\,,1\,,\dots$.

References

1. M. Aigner, *Combinatorial theory*, Springer, New York, 1979.
2. M. Berger, P. Gauduchon, and E. Mazet, *Le spectre d'une variété riemannienne*, Springer, New York, 1971.
3. P. Cassou-Noguès, *Dirichlet series associated with a polynomial*, Lecture Notes in Phys., vol. 47, Springer, New York, 1990, pp. 244–252.
4. P. Gilkey, *Invariance theory. The heat equation and the Atiyah-Singer index theorem*, Publish or Perish, Cambridge, MA, 1984.
5. I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series and products*, Academic Press, New York, 1980.
6. H. Mellin, *Eine Formel für den Logarithmus transzendenter Funktionen von endlichem Geschlecht*, Acta Soc. Sci. Fennicae **29** (1900), 3–49.
7. S. Minakshisundaram, *Zeta functions on the sphere*, J. Indian Math. Soc. **13** (1949), 41–48.
8. Minking Eie, *On a Dirichlet series associated with a polynomial*, Proc. Amer. Math. Soc. **110** (1990), 583–590.

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