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A HOMOGENEOUS, GLOBALLY SOLVABLE DIFFERENTIAL OPERATOR ON A NILPOTENT LIE GROUP WHICH HAS NO TEMPERED FUNDAMENTAL SOLUTION

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Dedicated to the memory of Lawrence Corwin

ABSTRACT. We present an example of a homogeneous, left-invariant differential operator on the Heisenberg group H_3 which admits fundamental solutions but no tempered ones. This answers a question raised by Corwin in the negative.

Assume that N is a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$, and let $L \in \mathfrak{U}(\mathfrak{n})$ be a left-invariant differential operator on N. If N is abelian, any such L can be considered as a constant coefficient differential operator on some Euclidean space $\mathbb{R}^n$ and, by the theorem of Malgrange and Ehrenpreis (see [H2]), has a fundamental solution $F \in \mathscr{D}'(\mathbb{R}^n)$, i.e., $LF = \delta$, where δ denotes the point measure at the identity. In fact, it was proved later by Hörmander [H1] and Lojasiewicz [L] that one can even find a tempered fundamental solution $F \in \mathscr{S}'(\mathbb{R}^n)$.

The situation becomes drastically different if N is nonabelian, since then there exist many operators in $\mathfrak{U}(\mathfrak{n})$ which are not even locally solvable. Assume in the sequel that N admits a one-parameter family $\{\delta_r\}_{r>0}$ of automorphic dilations (see [FS]) and that L is homogeneous, i.e., $L(\varphi\circ\delta_r) = r^m(L\varphi)\circ\delta_r$ for some $m > 0$ and every $\varphi \in \mathscr{D}(N)$, $r > 0$. Then it is at least true that various notions of solvability coincide for L. For instance, L is locally solvable at some point of N if and only if $LC^\infty(N) = C^\infty(N)$, if and only if L has a fundamental solution $F \in \mathscr{D}'(N)$ (see, e.g., [B, M1]). Moreover, if L^T, the transpose of L, is hypoelliptic, then the same is true of the operator LL^T, as can be seen by Helffer-Nourrigat's theorem [HN], and one can make use of the homogeneity of L in order to prove that LL^T, hence also L, has even a tempered fundamental solution [F, G].

So, a natural question, which seems to have been open hitherto, is whether any solvable, homogeneous, left-invariant differential operator or, more generally, every globally solvable left-invariant differential operator on a nilpotent Lie group has a tempered fundamental solution. In the latter form, this question was raised by Corwin in [C].

The purpose of this note is to present an example on the 7-dimensional

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Heisenberg group H_3 which answers this question in the negative, even for homogeneous operators.

Namely, if $X_1, X_2, X_3, Y_1, Y_2, Y_3, U$ denotes the standard basis of the Lie algebra $\mathfrak{h}_3$ of H_3 (with nontrivial brackets $[X_j, Y_j] = U$, $j = 1, 2, 3$), we set

$$L = (X_1^2 + Y_1^2) - \lambda(X_2^2 + Y_2^2) + Y_3^2, \qquad \lambda \in \mathbb{R}\backslash\{0\}. \tag{1}$$

Adopting the notation used throughout [MR1, MR2, M2], we have $L = \triangle_S$, where $S \in \mathfrak{sp}(3, \mathbb{R})$ is given by the matrix

$$S = \begin{pmatrix} H & 0 \\ 0 & N \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \lambda \\ 1 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2}$$

with respect to the basis $X_1, X_2, Y_1, Y_2, X_3, Y_3$. Since S is not semisimple, L is locally solvable for every $\lambda \in \mathbb{R}$ by [MR2, Theorem (i.3)].

Now assume there is some $F \in \mathcal{S}'(H_3)$ such that $LF = \delta$. If (z, u), with $z \in \mathbb{R}^6$, $u \in \mathbb{R}$, denote the usual coordinates of H_3, we define as in [MR1]

$$\mathcal{F}_u(z, \mu) := f^\mu(z) := \int_{\mathbb{R}} f(z, u) e^{-2\pi i \mu u} du, \qquad \mu \in \mathbb{R},$$

for $f \in \mathcal{S}(H_3)$. This partial Fourier transform turns L into the "twisted" differential operator $\tilde{L}^\mu$ given by the formula

$$(Lf)^\mu = \tilde{L}^\mu f^\mu, \tag{3}$$

if $f \in \mathcal{S}(H_3)$. Let $\delta_r(z, u) = (rz, r^2 u)$ denote the usual dilations on H_3, and fix a real function $\chi \in C_0^\infty(\mathbb{R}^+)$ with support contained in the interval [1, 2] and $\int \chi(r)\, dr = 1$. For $\varphi \in \mathcal{S}(\mathbb{R}^6)$ and $j = 0, 1$, we set

$$A_j\varphi(z, u) := \int_0^\infty \varphi(r^{1/2} z) e^{-2\pi i r u} \chi(r) r^j \, dr.$$

Then A_j defines a continuous linear operator from $\mathcal{S}(\mathbb{R}^6)$ into $\mathcal{S}(H_3)$, a fact which follows easily from the formula

$$A_j\varphi = \mathcal{F}_u(E_j\varphi),$$

where $E_j\varphi$ is defined by

$$E_j\varphi(z, \mu) := \varphi(\mu^{1/2} z)\chi(\mu)\mu^j.$$

Moreover, from (3) one easily sees that

$$L\mathcal{F}_u(E_0\varphi) = \mathcal{F}_u E_1(\tilde{L}^T\varphi);$$

hence

$$LA_0\varphi = A_1\tilde{L}^T\varphi, \tag{4}$$

where we have set $\tilde{L} := \tilde{L}^1$.

Let $A_1^T : \mathscr{S}'(H_3) \to \mathscr{S}'(\mathbb{R}^6)$ denote the adjoint operator to A_1, and let $K \in \mathscr{S}'(\mathbb{R}^6)$ be given by $K = A_1^T(F)$. Then, by (4),

$$\begin{aligned}\langle \tilde{L}K, \varphi\rangle = \langle K, \tilde{L}^T\varphi\rangle = \langle F, A_1\tilde{L}^T\varphi\rangle = \langle F, LA_0\varphi\rangle \\ = \langle LF, A_0\varphi\rangle = (A_0\varphi)(0),\end{aligned}$$

since $L = L^T$. Moreover, since

$$(A_0\varphi)(0) = \varphi(0)\int_0^\infty \chi(r)\,dr = \varphi(0),$$

we see that $\tilde{L}K = \delta$, i.e., we have proved the following

Lemma 1. *Assume L (given by (1)) has a tempered fundamental solution. Then the same is true of $\tilde{L}$.*

Finally, we can invoke [M2] in order to prove

Proposition 2. *Let L be given by (1). Then:*

(i) *L has a fundamental solution $F \in \mathscr{D}'(H_3)$ for every $\lambda \in \mathbb{R}$.*

(ii) *If L has a tempered fundamental solution $F \in \mathscr{S}'(H_3)$, then there are constants $C > 0$, $r \in \mathbb{N}$ such that*

$$|\lambda - p/q| > Cq^{-r} \tag{5}$$

whenever p and q are odd positive integers such that $\lambda - p/q > 0$. In particular, L has no tempered fundamental solution, if $\lambda = \lambda_0$, where $\lambda_0 := \sum_{k=0}^\infty 3^{-k!}$.

Proof. It has been shown in [MR2, Proposition 3.9] that the Liouville number λ_0 violates condition (5), so there remains only to prove (ii).

But, in the notation of [M2], the matrix S associated to L is of type $(E1)$, and condition (5) is equivalent to [M2, Theorem 1.1, condition (1.8)]. Therefore, by [M2, Corollary 3.2 and Theorem 1.1], $\tilde{L}$ can have a tempered fundamental solution only if (5) holds; hence (ii) follows from Lemma 1. Q.E.D.

Remark 3. In [M1] we showed that a homogeneous operator $L \in \mathfrak{U}(\mathfrak{n})$ is not locally solvable if there is a sequence $\{\psi_j\}_j \subset \mathscr{S}(N)$ such that $\psi_j(0) = 1$ for every j and

$$\lim_{j\to\infty} \|\psi_j\|_{(N)}\|L^T\psi_j\|_{(N)} = 0 \tag{6}$$

for every Schwartz-norm $\|\cdot\|_{(N)}$. This condition relaxes the necessary condition for local solvability in [CR] and was crucial in [MR2] but may look somewhat unnatural. One is tempted to ask if (6) could be replaced by

$$\lim_{j\to\infty} \|L^T\psi_j\|_{(N)} = 0. \tag{7}$$

However, Proposition 2 implies that this is not possible, for, if we could replace (6) by (7), then local solvability of L would imply an estimate of the form

$$|\psi(0)| \le \|L^T\psi\|_{(N)}, \qquad \psi \in \mathscr{S}(N),$$

for some Schwartz-norm $\|\cdot\|_{(N)}$. And, by the Hahn-Banach theorem, this would mean that L had a tempered fundamental solution.

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